EnLeMaH

Guidelines for Enactive Mathematical Learning at Home

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1. Theoretical Framework for an enactive teaching-learning practice

1.1. Introduction

Introducing an enactive approach to Mathematics teaching helps students build a mental network to understand mathematical concepts and relations and how they can use Mathematics in their daily lives and in their jobs. Enactive learning means having handmade activities, experiments and concrete handling with material to enter new mathematical topics, have mental representations of mathematical content and discover mathematical relations.

Consequently, enactive methodologies help to increase the understanding and the attractiveness of Mathematics [...] and, to a broader extent, contribute to reduce underperformance. Nevertheless, the adoption of an enactive approach to Mathematics is based on two main preconditions or premises

On the one hand, teachers need to acquire and be equipped with the adequate pedagogical skills to implement this methodology, particularly when it concerns to its applicability to the context of digital education and training. On the other hand, enactive materials can be hard to obtain in the current context strongly affected by the COVID-19 crisis and especially considering the fact that, for several months, schools were closed and education was distance-based.

EnLeMaH seeks to promote the adoption of innovative digital pedagogical competencies for Mathematics school teachers, which will enable them to develop the knowledge and skills to:

a) Implement an enactive teaching & learning methodology adapted to the context of digital education that contributes to make Mathematics more attractive for school students (12-16 years);
b) Guide school students in creating, using household supplies, enactive materials that support their learning processes, with a special focus on Mathematics learning in the field of functions.

To this purpose, EnLeMaH distinguishes three main phases in the Project. A first phase (Intellectual Output 1) is defined by the creation of a theoretical framework on which to subsequently base the creation of the EnLeMaH course for secondary school teachers of mathematics. The second phase of the project (Intellectual Output 2) consists of the creation of the learning units and materials for the teacher-training course. The third phase (Intellectual Output 3), parallel to the second phase, is the

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structuring of these learning units in the form of an online course, the EnLeMaH Teachers Training Course.

Thus, this document "Guidelines for enactive learning at Home" (Intellectual Output 1), gathers:

- a theoretical approach to enactivism based on the authors Maturana, Varela, Brown and on the theory of cognitive growth including modes of representation of Bruner
- its implications for the teaching/learning of mathematical contents from an enactive perspective;
- a review of the mathematical curricular contents (12-16 years) in the different partner countries of this project;
- a summary of the tradition that each of the countries of the project has with enactive approaches in the teaching of mathematics and, finally,
- the criteria that EnLeMaH establishes to be able to classify an activity as enactive. Together with these criteria, EnLeMaH proposes a template to follow in the creation of new enactive learning activities.

1.2. Principles for researching as an enactivist

In this chapter, we take a look at the relation between researching and teaching. For the purpose of the EnLeMaH-project aspects are fundamental for the understanding of learning and teaching in enactive situations. Our approach for collaborating for enactive ideas is using the principles for researching as an enactivist by Brown (2015). In this section, we will introduce his three principles.

1.2.1. Principle 1: Implications of knowing/ doing and '-ing's

Enactivist theories are developed out of the biological basis of being and the enactive approach consists of two points (Brown, 2015):

- Perception consists in perceptually guided action and
- Cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (Varela et al. 1991, pp. 172–173)

Thus, "knowing is doing is learning is being". We are, literally, what we do, our environment having created us as we have created our world. (Brown, 2015).

Figure 1: Collaborative Groups Sharing Experience

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> This is not an individual cognitive structure. We are co-emergent and where there is a coordination of actions, like in a classroom, or a collaborative group in a research project, a culture of practices emerges that is *good-enough* (effective action) to get done what needs to be done. (Brown, 2015, p.189).

Then we, as a collaborative group of researchers and teachers, can share our own experiences about learning and teaching, allowing us to write and discuss to obtain final conclusions (figure 1).

1.2.2. Principle 2: history of structural coupling

Maturana and Varela (1992, p. 75) talk about "structural coupling whenever there is a history of recurrent interactions leading to the structural congruence between two (or more) systems". That is, a need expressed by another triggers the action of helping to fulfill my own needs.

How can we use the idea of structural coupling in collaborative groups?

- There are no certainties in teaching and learning, but…
- What we experience through our actions is an interpretation, so…
- "Two people cannot see the same thing nor share the same awareness. However, we can communicate because we can talk about the details of common experiences and, in doing so, the gap between interpretations can be reduced" (Brown, 2015, p.189).

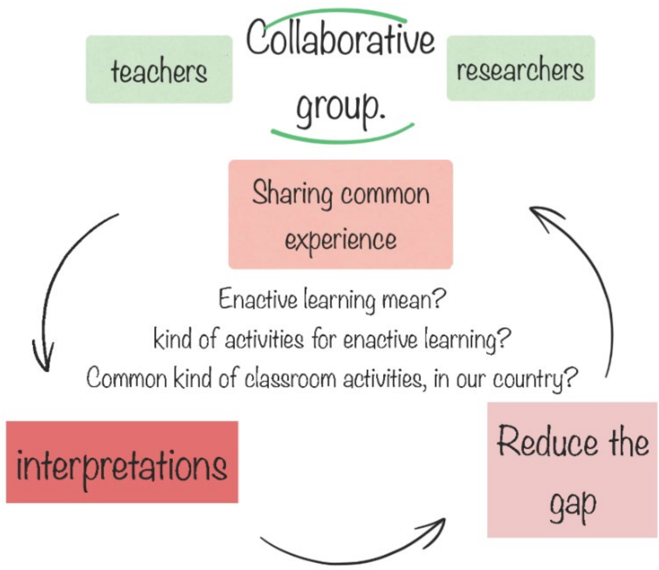

Figure 2. Collaborative group in enactive learning

So when we are able to share our experiences and discuss on the basis of the necessary questions for our project, the gap between our interpretations narrows and gives way to a common vision.

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1.2.3. Principle 3: multiple perspectives and Principle 4: effective action

"The importance of working from and with multiple perspectives, and the creation of models and theories which are good-enough for, not definitively of" (Reid 1996, p. 207). *Effective'* is a technical word in enactivist ideas, linked to cognitive structures and learning '*effective'* actions was used as 'good-enough for' the learning of mathematics of the children and for supporting the new teachers in reflecting and researching.

The members of the collaborative group of researcher/teachers are not working towards 'answers' but to extending our range of possible effective practices. In the following chapters, this range will be more precise in the field of representations and the biological basis for enactive learning.

1.3. Theory of Cognitive Growth by Jerome S. Bruner

> It is fruitful, I think, to distinguish three systems of processing information by which human beings construct models of their word: through action, through imagery, and through language (Bruner 1965, p.1).

As Bruner stated above, individuals represent their learning and the world in which they live through action if they cannot do so using images or words. He assumed about learning that any subject could be taught at any stage of development in such a way that it's cognitive abilities were met. To learn a more highly skilled activity, it has to „be decomposed into simpler components, each of which can be carried out by a less skilled operator"(Bruner 1964, p. 2). Representations are the end product of a system of coding and processing past experiences. Hence, he introduced a model including three modes of representation as included in table 1). He believed that people represent their knowledge in those three ways. Modes of representations "are not structures, but rather involve different forms of cognitive processing" (Schunk 2012, p.457).

Table 1. Modes of representations

Name of the Mode of Representation	Description	Examples in Mathematics Class
Enactive Mode of Representation	Suggests that individuals represent their learning and the world in which they live through action if they cannot use images or words. A student best understands their environment by interacting with the objects around him	Using material to represent a mathematical concept. 

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Iconic Mode of Representation		Summarizes events by the selective organization of precepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images.	Using images (e.g. pictures of the (mathematical) situation, graphs) to represent a mathematical concept
Symbolic Mode of Representation	Verbal-symbolic	Each word has a fixed relation to something it represents	Using (actually spoken) word to represent a mathematical concept
	Non-verbal-symbolic	Each symbol has a fixed relation to something it represents	Using written sentences and mathematical symbols (e.g. equations) to represent a mathematical concept

For enactive learning, these modes of representation correspond in the learning process (figure 3). Enactive situations will be translated into iconic representations and reflect back to that situation. The basis of the enactivity and/ or the iconic representation will be the basis of symbolic transformation and the other way around: symbolic modes will be discovered in iconic and enactive modes.

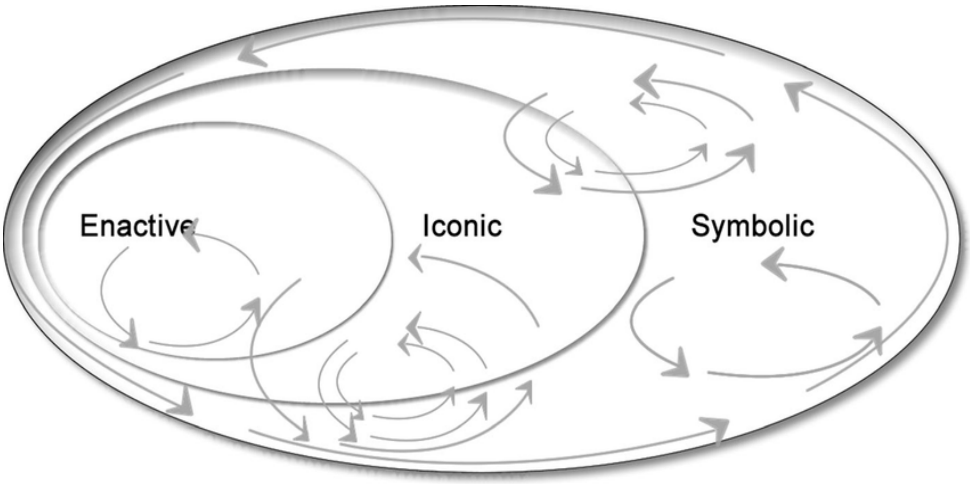

Figure 3. Enactive, iconic and symbolic as nested, co-implicated and simultaneous (Francis, Khan, David, 2016, p.8)

By this, the three modes of representation deal with a central theoretical aspect for the EnLeMaH-project: For the understanding and the creation of enactive learning activities the separation between different modes are basic. In the next chapter, we take a deeper look at aspects for designing enactive learning.

1.4. Enactive learning: Biological bases

According to Di Paolo (2018), the term enactive was used prior to the biological bases that shape the theory today. For example, Bruner (1966), used the term enactive to establish a relationship between representations and bodily aspects belonging to a person's lived experience. Currently the meaning of enactivism is based on the works started by the biologist Francisco Varela (1946-2001) and on the works carried out jointly with Maturana (1980, 1987). Today this theoretical perspective continues its development by various groups of researchers who are focused on different areas of study: Brown, L., 2011, 2012, 2015. Coles, A., 2013, 2015. Di Paolo., 2005, 2018. Lozano, MD. 2004, 2015, to name a few.

Varela, Thompson and Rosch (1991) used the words ‘enaction’ and ‘enactive’ to describe the non-representation list view of cognition they set out, cognition as “embodied action”. This refers to two important points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided (Varela et al. 1991, pp. 172-173).

Therefore to understand what enaction is, we must understand what perception is. It is important to note that perception is sometimes seen as a passive process (e.g. when light enters your eyes and you are able to create an image), but in enactivism, perception is an active process, and without action there is no perception. On the other hand, this active process is determined by the structure of the perceiver, for example: how a bird perceives a certain situation is very different from how a person might perceive it.

On one side, the *organization* is understood as the relationships that must exist between the components of something to be recognized as a member of a specific class, and on the other hand, *structure* of something is understood as the components and relationships that specifically constitute a particular unit that its organization carries out (Lozano, 2014). Maturana and Varela point out that living beings are systems in which the structure is continually changing, but whose organization is preserved (Maturana 1998a in Lozano, 2014). This occurs in the particular mode of organization they call *autopoiesis*. Therefore, an autopoietic system is one that despite being constantly changing and producing new reference systems, the result will always be the same producer. According to Matura (1987), the problem would be how to handle the problem of change of structure and show how an organism that exists in an environment and that operates adequately according to its needs, can undergo continuous structural changes even if the environment is changing. So, this could be an approximation to the problem of teaching mathematics, a mathematics student is a system that internally organizes itself at every moment. So, every time a stimulus reaches it, (for example a mathematical symbol), it is immediately incorporated into the student's structure, into her being.

On the other hand, according to Lozano (2014), when living beings interact with the environment where other living beings are included and there is a recurring interaction between two systems, then both will change in a similar way. From this perspective, we could say that when a math student interacts repeatedly with her teacher and with

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the other students, together they will create a history of interactions. Therefore, the structure of all those who are participating in these classes can change in a similar way, creating new forms of communication and work. If this does not happen, then the structural changes do not lead to adaptation to the environment. Lozano (2014), presents a clear example for this: if a student repeatedly fails math tests, in a certain context this could mean that the student changes the study group she is in.

Something important to mention, is that the world is not something that is given to us, but something that we relate to by moving, touching, breathing, and eating, this is what Maturana and Varela called cognition as enactive (Maturana and Varela, 1992). So, enactivism indicates that our mental activity (thoughts, images, emotions) is rooted in the actions we carry out with and through our bodies. The enactivism point of view, learning arises as we actively interact with the environment, so it cannot be thought of as absorption of information and cognition is not a phenomenon that arises within the head or body of a single individual, but arises from continuous interactions with the environment, which in turn is modified by these. In our case, society and culture are part of our environment as human beings.

This conception of enactivism from a biological point of view, invites us to think about the importance of the type of activities that are selected to deal with a content that has a learning objective in mathematics. In general, there are many materials that we can have at our disposal, but we must take into consideration the context in which our students develop, the type of environment and the type of structure that composes them. That is, we must try to create models of tasks appropriate to the level of our students along with the use of materials that allow them to use the body as the main tool for the development of new learning, and so the images that students create about the activity, can be experienced by themselves through their own actions. All this must be accompanied by strategies that allow a good development of class activities and good management of the environment considering the individual and group characteristics.

Therefore, the main goal of the activities that should be produced in the framework of this EnleMaH project should respect the biological vision of enactivism i.e., start with learning objectives that can be achieved through actions involving the body. Then with the same images and representations that students will create through the activity, we can rely on Bruner’s theory to make the necessary transformations for the student to link his own visualizations to other modes of representation, without losing sight of the fact that the generation of the student's individual representations is given by his own perception of the actions performed.

In the following section we can find a small guideline on strategies that can be used when selecting an enactive activity and that also uses Bruner for the comprehension of conceptions and misconceptions with the help of the different modes of representation.

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1.5. Principles and enactive ideas for designing activities

1. Starting with an (closed) activity (which may involve teaching a new skill).
2. Considering at least two contrasting examples (where possible, images) and collecting responses on a ‘common board’.
3. Asking students to comment on what is the same or different about contrasting examples and/or to pose questions.
4. Having a challenge prepared in case no questions are forthcoming.
5. Introducing language and notation arising from student distinctions.
6. Opportunities for students to spot patterns, make conjectures and work on proving them (hence involving generalizing and algebra).
7. Opportunities for the teacher to teach further new skills and for students to practice skills in different contexts.

The activity will, where possible, involve something visible or tangible and which all students can do. The challenge and opportunity to teach skills in different contexts are linked to the power humans have of extraction. When we do something in different contexts, it is more likely we can extract the skill and retain it for use another time. (Coles and Brown 2013, p. 186–187). At least one possible approximation of enactive learning definition: “learning is seeing more, seeing differently, in a recursive process linked to *actions in the world* giving feedback leading to adapted actions until the behaviors become effective, that is, no perturbations are created and the action is therefore ‘good-enough-for’ being in the world” (Brown, 2015, p.190). So far we have reviewed the biological bases of enactive learning, and figure 4 shows us a possible learning context. In the next section we will see how the enactive bases can be linked to the modes of representation proposed by Bruner, who takes enactivism as the basis to show how knowledge starts to develop.

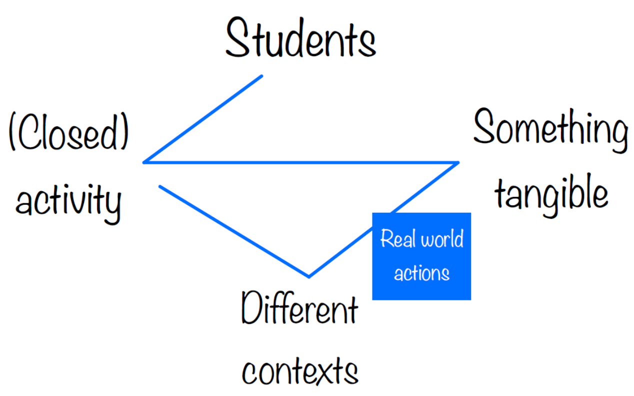

Figure 4.Learning context

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The theoretical input in the first three chapters dealing with different points of views towards enactive learning: the connection between researcher and teacher, the kind of representation mode, and aspects for designing enactive learning. In the following chapter a special kind of activity will take into account: the experiment as a central activity for enactive learning.

1.6. Experiments

An experiment is a scientific method by which information shall be gathered. It is used in school as well as in university and also in multiple subjects. But there is a difference between experiments in mathematics and other subjects, which can be seen in the first paragraph. Hence, mathematical experiments have two different purposes. The separated steps of mathematical experiments can be seen in the second paragraph.

1.6.1. The method “experiment” - differences between subjects

Following Kirchner et al., there are different purposes for experiments in natural sciences: gathering knowledge, demonstration of phenomena, giving ‘primary experiences’ to pupils or the verification of a relation or model (cp. Kircher/Häußler/Girwidz, 2009). All these purposes lead to a better understanding of nature. Typically, there are up to six steps for an experiment in natural sciences. At first the object of investigation must be clarified. Afterwards pupils have to collect hypotheses as a second step. The third and fourth steps are the planning and execution of the experiment. Whilst execution the measurement of data is important in order to analyze these data for correlation between quantities. This analysis is the fifth step and is only followed by the last step: the interpretation of results. In the last step the results and hypotheses are compared (cp. loc. cit.). The interpretation of results itself often leads to another object of investigation and thus to another experiment. Even though there are differentiations between experiments, e. g., ‘Will pupils or the teacher execute?’ or ‘In what phase of the lesson is the experiment integrated?’ (cp. Wiesner/Schecker/Hopf, 2017, p. 106-114), in natural sciences every experiment is about real objects.

There are similarities and differences between mathematical experiments and experiments in natural sciences. In both subjects an experiment describes a way of gathering knowledge by observation of controlled action with ‘objects’ (cp. Ludwig/Oldenburg, 2007, p. 4). The process of experimenting in mathematics is largely identical with the process in natural sciences. Though it is not necessary to collect hypotheses before trial. Examining several examples or handling with material is a starting point for pupils to build hypotheses, so step two can be replaced after step three and four. As mathematical facts need to be proved the sixth step of interpretation suggests approaches to a formal proof or leads to a repetition of the experiment with slightly different conditions (cp. Philipp, 2012, p. 27 or Goy/Kleine, 2015, p. 6). Finally

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mathematical experiments can be detached from real objects (cp. paragraph 4). Thus, experimenting in mathematics needs the pupil to know heuristics and teaches process-oriented competences.

1.6.2. Mathematical Experiments as a process

As mentioned before, experimenting in mathematics is a cycle of different steps. Referring to Philipp (2012) and Goy/Kleine (2015) there are four main steps for mathematical experiments:

1. Stating the mathematical problem/question
2. Generation of hypotheses
3. Planning, execution and analysis of the experiment (short: ‘trial’)
4. Elaboration of a mathematical model, concept or proof

For every experiment stating the mathematical problem or question is the first and the elaboration of a model, concept or proof is the last step. The order of the other steps can be changed considering the experiment’s aim. If the experiment is about to verify or falsify hypotheses, those hypotheses have to be generated first. If the experiment aims on pupils learning how to experiment or making up their own models and concepts, the trial has to be placed before the generation of hypotheses (cp. Goy/Kleine, 2015, p. 5f).

Heintz constructs three contexts for mathematical experiments: discovery, validation and persuasion (cp. Philipp, 2012, p. 25). Context of discovery pertains to the generation of hypotheses and is meant as systematic trial in order to explore unidentified relations. Here knowledge is obtained by induction. Otherwise, knowledge is obtained by deduction when a given hypothesis is validated by the mathematical experiment. In this case the experiment is set in the context of validation. Finally, if neither discovery nor validation is needed because a relation, concept or model is already confirmed there is another context for mathematical experiment: persuasion. In this case the experiment shall convince the students (cp. loc. cit.).

Based on the theoretical background, enactive learning at the EnLeMaH-project can be described as hand-based activities, which enables students to discover mathematical relations or prove mathematical connections. The different phases of an experiment can be a guideline for teachers on the basis of the designing principles, to arrange an enactive learning situation.

To do so, regarding the idea of enactivism and basic concept of experiments in Mathematics, we will further take a look at the school curricula of each country that is integrated in the development of EnLeMaH-project, to ensure the context made transfers well to the corresponding school system.

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2. Short curricula summary

In the EnLeMaH-project researches and teachers from four different countries take part. In the purpose of this project, the focus is on creating enactive learning situations in the field of functions for students aged 12 to 16 years old. In the following chapters the different curricula in the countries will take into account, to found a basis for enactive learning in this field of mathematics at school.

2.1. Curricula for functions in Croatia

In the Croatian mathematical curricula there are five main topics: (A) Numbers, (B) Algebra and functions, (C) Geometry, (D) Measuring and (E) Statistics and probability. In the domain of “Algebra and functions” pupils define functions and interpret, compare, graphically represent and learn about their properties by recognizing regularities and describing the dependence of two quantities in the language of algebra. They model situations by describing them algebraically and solve real-life problems involving regularities or functional dependencies. The following table gives an overview about different kinds of functions for 12 to 16 years old students.

Table 2: Croatia's Curricula

Educational outcomes		**Concept of function**	**Graphical representation of function**	**Proportion**	**Linear function**	**Quadratic function**
PRIMARY EDUCATION	**6th grade**				Solve and apply a linear equation	
	7th grade	Additional content: Connect linear dependence with linear function	Linear dependence	Proportionality and inverse proportionality	Solve and apply a linear equation Apply a linear dependence Additional content: Solve simple linear inequality	
	8th grade				Solve and apply a linear equation Solve and apply a system of two linear equations with two unknowns	Solve and apply a quadratic equation of the form $x^2 = k$

SECONDARY EDUCATION	**1st grade (14,15 years old)**	Linear function	Graph of a linear function	Apply proportionality and percentage	Apply a linear equation and a system of linear equations Apply linear inequalities in problem situations Connect different representations of a linear function Apply s a linear function in solving problems	
	2nd grade (15,16 years old)	Concept and analysis of a function	Analysis of the graphical representation of a function			Solve and apply a quadratic equation Apply a quadratic function

To take a deeper look into those grades and their demand:

6th and 7th grade: Pupils analyze a problem situation for the sets Z and Q and formulate it as a linear equation. In the 6th grade (also 7th grade) they reduce a more complex linear equation to form $ax = b$ $(ax + b = 0)$ for a and b being non-negative rational numbers (rational numbers) by applying equivalences of equations. Also, they solve a simple linear equation with absolute value. Besides solving linear equations, pupils reconsider the accuracy and meaningfulness of the obtained solution and explain it in the context of the problem. An additional content is to solve simple linear inequality. In real life situations pupils recognize and explain proportionality and inverse proportionality and determine and interpret the coefficient of proportionality and inverse proportionality. Also, they connect the coefficient of proportionality with the ratio of two proportional quantities. It is a recommendation to encourage pupils to use an intuitive approach to solve problems of proportionality and inverse proportionality. Also, to determine and explain some complex units of measurement (km/h, m/s, g/cm^3, kg/m^3, residents/km^2) and to convert currencies.

In the “Apply a linear dependence” outcome teachers do not test the pupil’s calculation technique, but their logical thinking and their ability to analyze the problem. Pupils explain linear dependence of quantities from real life problem situations. The emphasis is on the study of dependent quantities, on the translation of the observed situation of linear dependence into an algebraic notation, on the interpretation of a graphical representation of linear dependence and on the analysis of change. Pupils form a table

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of associated values of linearly dependent data. They graphically represent linear dependence and make a connection between linear dependence and linear function. *8th grade*: In given problems pupils recognize the possibility of solving a system of two linear equations with two unknowns. If the system is more complex, they reduce it to a standard form and solve it by a given or arbitrary method. Besides that, they discuss the existence of the obtained solution (uniqueness, non-existence, infinitely many solutions). Pupils describe a quadratic equation of the form $x^2 = k$, where k is a nonnegative rational number and they apply the quadratic equation to solve problem situations and for the purpose of representing quantities by mathematical formulas. They interpret the existence of two solutions. *1st grade:* Pupils write a given problem in the form of a linear equation or system of linear equations and solve them. They discuss the existence of its solution considering the value of parameters. Pupils solve linear inequalities by writing the solution in different ways, systems of linear inequalities with one unknown and simple linear inequalities with absolute value. Pupils connect different representations of a linear function. For a given linear function, they calculate the value of the function, draw a graph, define and determine the zero of the function, and interpret the coefficients.

Also, they represent the linear function by tables and graphs and describe the influence of coefficients on the position of the graph. Pupils read arguments and values from the graph and determine the coefficients and the function. From the given elements (arguments and values, graph points, coefficients) they determine the function. Extended content is to draw a graph of the absolute value function. From the given data, pupils write the linear dependence as a linear function. In problem situations, they recognize a linear dependence, write it down as a function and apply it to analyze the problem. They have to analyze a problem from its graphical representation. For example, *Design a task shown by the given graph (Figure 5):*

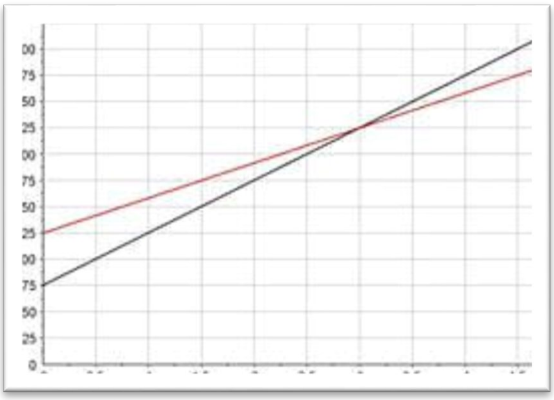

Figure 5: Designing task from a given graph

2nd grade: Pupils effectively solve a quadratic equation, check its solutions and argue the nature of the solutions. For example: "*Without solving the equation* $3x^2 + 4x - 1 = 0$ *determine the nature of its solution".* Also, they apply discriminant in determining the nature of the solution of a quadratic equation. Furthermore, they solve equations

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that reduce to a quadratic equation and model a problem situation and determine solutions.

In the domain "*Analysis of a function*" pupils calculate the functional value of a polynomial, rational and irrational function. They explain the concept of a function and computationally determine the domain and codomain of simple rational and irrational functions. Also, they define bijection and recognize it on the examples of sets shown by Venn diagrams and determine the image of linear and quadratic function. The simple rational functions are of form $f(x) = \frac{a}{bx+c}$. The simple irrational functions are of form $f(x) = \sqrt{ax+b}$.

In the domain "*Analysis of the graphical representation of a function*" pupils display functions graphically and determine the domain, codomain and image of a function by using its graph. For example: "*Draw graphs for $f(x) = \frac{1}{x}$ and $f(x) = \sqrt{x}$ determining the functional value for some values of the variable x. Then, sketch graphs of their inverse functions by mapping the functions over the line $y = x$.*" Pupils draw a graph of a quadratic function with rational coefficients. They determine the zero, the intersection with the ordinate, the vertex of the parabola, the axis of symmetry and also the course of the function. They solve simple quadratic inequalities. When graphically representing a quadratic function, pupils explain the shape of the function depending on the discriminant and the leading coefficient.

Extended content is to specify a function from the graph. For example: Pupils will graphically represent a function of the form $f(x) = a(x - x_0)^2 + y_0$ by using translation and a function of the form $f(x) = ax^2 + bx + c$ by the method of five points (real zero, vertex of the parabola, intersection with ordinate, mapping the intersection over the axis of symmetry. The problem situation includes problems with extremes and determining the intersections of quadratic and linear functions. For example: "*By tracking the sales of a product, it was found that sales can be described by a quadratic function $f(x) = -\frac{3}{20}x^2 + 12x - 180$, where x is the price of the product, and $f(x)$ is the number of pieces of the product sold at the price of x. How many products will be sold if the price is 30 kunas? How much will the trader earn? At what price will sales of this product be maximal?*"

The topic of functions gets more and more important during the grades. At the beginning the numbers are the main topic for mathematical education at school with approx. 50% of learning time. Functional thinking is 20% of the learning time. At the end of the period dealing with functions is the main topic at Croatian schools with 45% of learning time.

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2.2. Curricula for functions in Germany

In the ordinary school system in Germany, there exist three kinds of graduation in secondary education up to 16 years old. They are all dealing with the concept of functions and different types of functions. There will be an overview in the following table.

Table 3: German's Curricula

Type of function **Level of graduation**	**Concept**	**Proportion**	**Linear**	**Quadratic**	**Trigonometric**	**Exponential**
„Hauptschulabschluss" (low level) Represents graduating 9th grade (14,15 years old)						
„Mittlerer Schulabschluss" (medium level) Represents graduating 10th grade (15,16 years old)					$\sin(x)$ not in general	$a \cdot b^x$ not in general
„Zulassung Oberstufe" (high level)					$a \cdot \sin(bx + c) + d$	$a \cdot b^x$

In German curriculum the field of functions is deeply connected with the field of equations. Primary education mostly ends with grade four.
In secondary education you address the field of functions in the following increasing contents:

- grade 5 and 6: pre concepts of relations between quantities; use of easy terms to describe situations; solving easy real situation problems with methods like systematical trying.
- grade 7 and 8: concept of relations between numbers and quantities; proportional and antiproportional relations; percentage calculation as a concept of proportional relation; concept of term; concept of linear equation in easy situations (e.g. 2x+5 = 3) with different methods to solve equations like systematic trying, operating methods, and transformation. concept of function as a special relation; proportional and linear functions; concept of linear equations and solving them with algebraic and graphical

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methods; connection between function and equation; system of linear equations.

- grade 9 and 10: quadratic functions; variation of parameters; concept of quadratic equations and solving them with algebraic and graphical methods; connection between function and equation; ratio and easy equations with ratio. *optional:* system of functions (linear, quadratic) and solving them graphically and algebraically; other types of functions contrasting linear and quadratic functions like exponential functions in easy contexts; the sin x as an easy example of a periodic function.

In the curriculum there are two main topics: (1) calculus, (2) relations, functions, and equations. The fields of (3) plane geometry esp. with triangles and (4) statistics has not the importance comparing with (1) and (2). (5) Spatial geometry is at the end of the fields.

In grade 5 and 6, the field of relations, functions, and equations has an extension of approx. 20% of the mathematical curricula. Preconceptions are implemented in the field of quantities and plane geometry. In grade 7 and 8, this field has an extension of approx. 50%. It is the dominant topic in these grades. Also in grade 9 and 10, this field has an extent of 50%-60%. Dealing with special equations will also take part in plane geometry (e.g. ratio in the theorem of intersecting lines).

2.3. Curricula for functions in Lithuania

In Lithuanian schools the field of functions will develop as shown in the following table.

Table 4: Lithuania's Curricula

	Understanding and using tables, graphs and formulas	**Application of function models and properties**	**Application of the coordinate method to describe geometric shapes and to study their properties.**	**Graphical solution of equations, inequalities and their systems**	**Graph transformation**
5-6 classes (11,12 years old)	To read (analyze) the dependencies between two quantities expressed in simple graphs or tables. To explain in your own words what the	To solve the simplest tasks of everyday content, where the two sizes are directly proportional. To provide examples of directly proportional			

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	numbers on the Ox and Oy axes show. To find the value of one quantity from the given graph or table when the value of another quantity is specified.	quantities, explain how to find the value of one when the value of the other is known.			
7-8 classes (13,14 years old)	To use tables, graphs and formulas describing the dependencies of two sizes, solving simple problems of practical and mathematical content. To explain the concepts of independent and dependent variables in your own words, to know how they are denoted. In simple cases, to determine from a graph, formula, or table how to find a value of one quantity when a value of another quantity is specified.	To rely on models and properties of direct or inverse proportionality, the property of proportion in explaining solutions to simple problems of various content. To remember that directly proportional quantities are related to the equation $y / x = k$, and inversely proportional to the equality $x \cdot y = k$, to give examples of quantities related to such dependencies. In the simplest cases, to apply the basic property of proportion. To understand how many points are needed to be selected to draw a sketch of a graph of direct and inverse proportionality. To compile and complete a partial table of values of direct and inverse proportionality when $x > 0$, to draw sketches of their graphs.			

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		To be able to check whether the point belongs to the function schedule.			
9-10 classes (15,16) years old	To combine different ways of expressing functions, to apply the properties of a function in solving simple problems of practical and mathematical content. To describe the concepts of the independent variable (argument) and the independent variable (function), to write their symbols. In simple cases, from a graph, formula, or table to determine which quantity is independent, which is dependent, to know how to find the value of one when the value of another is specified. To determine from the graph whether the dependence of the two quantities is functional. Provide examples of functions and non-functions. Explain how to check whether a point belongs to a function graph. From the graph to find the areas of function	To rely on the models and properties of direct or inverse proportionality, linear, quadratic function, the property of proportion in explaining the solutions of simple problems of various content. To recognize direct or inverse proportionality, linear, quadratic functions expressed in various ways, provide examples of quantities related to these functions. In simple cases, apply the basic property of proportion. To understand how many points need to be chosen to draw the formulas y = kx + b, y = k / x and y = x^2, y = ax^2 + bx + c, y = a (x - m)^2 + n; y = a (x - x^1) (x - x^2) graphs of functions, to know what the names of the graphs are. To explain how to write the expression of a function from a graph of linear and quadratic functions.	To plot shapes in the coordinate system, to draw a symmetrical shape with respect to the point and the line, to describe the position of the shapes in the coordinate system in pairs of numbers. To find the length of the segment, the coordinates of the midpoint of the segment, where the coordinates of the ends of the segment are known. To associate a pair of numbers with its image in the coordinate system. To indicate to which coordinate quarter the point belongs. In the coordinate system, mark a point symmetric	To solve systems of linear equations graphically approximately. To approximate graphically the equations f (x) = a, f (x) = g (x) and the inequalities f (x) a, f (x) ≤a, f (x) ≥a, where f (x) and g (x) are functions of direct, inverse proportionality, linear, quadratic, and a is a number. To explain the essence of the graphic method in your own words. To find the approximate solution of the system of linear equations from the graph. Based on the example, to explain how to find a solution to an equation or inequality with one unknown from a graph.	To perform transformations of the graph y = x2: stretching on the Oy axis (y = ax2), thrusts on the Ox and Oy axes (y = x2 + n and y = (x - m) 2), symmetry with respect to the Ox axis (y = - x2); associate graph transformations with changes in the formula y = x2. To know how to transform the graph of a function using the graph's thrust on the Ox and Oy axes, tensile (pressure), symmetry. For example, having the graph of the function y = x2 explains how to draw the graph of the function y = a (x - m)2 + n.

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	definition and values, intervals of increase, decrease, stability of function values, maximum or minimum value of the function. To know how to find with which argument values, a function acquires a particular value, function values are positive (or negative) when the function is expressed in a graph or formula		to a given line or point, check that the two figures are symmetrical about the origin of the coordinates, the Ox and Oy axes. To use an example to explain how to find the length of a segment, the coordinates of the midpoint of the segment when the coordinates of the ends of the segment are known.		

2.4. Curricula for functions in Spain

In the Spanish school system, regarding Compulsory Secondary Education (E.S.O), there are four levels leading up to graduation at 16 years old and starting at 12 years old.

Table 5: Spain's Curricula Part 1

	Age	
Primary Education	1st – 6th grade	6 – 12
Compulsory Secondary Education (E.S.O)	1st grade	12 – 13
	2nd grade	13 – 14
	3rd grade	14 – 15
	4th grade	15 – 16

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Baccalaureate	1st – 2nd grade	16 – 18

The Spanish curricula in all courses is distributed in five learning blocks: I. "Processes, methods and attitudes in mathematics", II. "Numbers and algebra", III. "Geometry", IV. "Functions" and V. "Statistics and probability".

The main topics covered in each course, and particularly the content related to the functions, can be found in the following table:

Table 6: Spanish's Curricula Part 2

1st grade	**12-13**
In general: Students design graphical representations to explain the process followed using technological means. They also calculate the value of numerical expressions by means of elementary operations and powers of natural exponent applying the hierarchy of operations. They apply divisibility criteria for divisibility by 2, 3, 5, 9 and 11 to decompose into prime factors. Besides this, pupils identify the maximum common divisor and least common multiple, as well as opposites and absolute values. Finally, they understand the arithmetical and geometrical meanings of the Pythagorean Theorem and apply it. Specifically in functions: Pupils work on the organization of data in tables of values, on the Cartesian coordinates as well as on the representation of points on a system of coordinate axes, in this sense they must identify different points from their coordinates. Additional content at this stage is the identification of direct proportionality links through the analysis of their table of values. Students use counterexamples when magnitudes are not directly proportional.	
2nd grade	**13-14**
Students solve operations with integers and work on exponentiation. They also work with the decimal and sexagesimal systems, as well as with fractions, for example by solving arithmetic problems with fractional numbers. The Pythagorean Theorem is also included in the curricula at this level, as well as the study of geometric bodies. In addition, students perform probability calculations and read statistical graphs. Specifically in functions: Students work on the concept of function, the dependent and independent variables as well as on the concepts and calculation of growth, decay, maxima and minima. Students use proportionality functions (y=mx); linear functions (y=mx + n) and constant functions (y=k).	

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3rd grade	14-15

Students use the properties of rational numbers in operations through appropriate calculation in problem solving and know and use algebraic language to express statements in algebraic form. Students solve everyday problems by means of first- and second-degree equations and systems of two linear equations with two unknowns.

Students identify and describe the characteristics of plane figures and elementary geometric bodies with their geometric configurations. In the topics dedicated to statistics, pupils learn to analyses information with data through appropriate tables and graphs with conclusions that represent the population studied, among many other aspects.

Specifically in functions:
Students continue to work on the concept of function, the graph as a way of representing the relationship between two variables (function), its nomenclature, as well as other basic concepts related to functions, such as the independent and dependent variables. Students at this level carry out the interpretation of given functions by means of graphs and the assignment of graphs to functions, and vice versa.

Students work on the variations of a function, its growth and decay and its maxima and minima. In this sense, they can determine growth and decay as well as maxima and minima of given functions by means of their graphs.

Discontinuity and continuity in a function, recognition of continuous and discontinuous functions, trend and long-term behavior are also explored, and students establish the trend of a function from a part.

The curricula also include the study of periodicity, the recognition of those functions that present periodicity and further work on analytical expression, as well as the assignment of analytical expressions to different graphs, and vice versa.

4th grade	15-16

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In this level, students work on the roots of a polynomial and factor it using Ruffini's rule. They also work on operations with polynomials and simple algebraic fractions and solve equations of second degree through factor decomposition and use basic trigonometry to solve problems. They are able to solve triangles using trigonometric ratios and they also work on the coordinates of points and vectors.

Pupils calculate areas, volumes of triangles, quadrilaterals, circles, etc. as well as the distance of a point and the modulus of its vector. Students also explain and represent graphically the linear, quadratic, inverse proportionality, exponential relationship.

Specifically in functions:
Students deepen in the concept of function, the graphical representation, table of values and analytical expression or formula; the relation of graphical and analytical expressions of functions; the domain of definition of a function as well as the restrictions to the domain of a function.
Students carry out the calculation of the domain of definition of various functions and analyze its characteristics, its discontinuity and continuity and deal with the construction of discontinuities, the growth, decrease, maxima and minima of a function, as well as the tendency and periodicity of a function.

In addition, the average rate of variation is studied at this level, as well as the average rate of variation of a function in an interval.

2.5. Conclusion

As shown in the chapters above, in the different participated countries of the EnLeMaH-project, the field of functions get the most important mathematical topic during the based learning period in secondary education. The different cultural background of the partners can assume that this importance will also exist besides the four countries. Moreover the shown curricula gives hints that the mathematical learning process in the different countries concerning functional concepts, functional types are similar in the way of order and arrangement. This is the basis to common work for teachers for enactive learning besides the country-specific points of view: the arrangement of learning functions at school are quite similar.

3. Enactive learning and teaching in EnLeMaH project

In this chapter the philosophy of enactive learning at school will take into account. Thereby the implementation of enactive learning in national standards, in teacher training programs and university programs will take into account.

3.1. Croatian philosophy

According to the Croatian Curriculum of the Course of Mathematics for Primary Schools and Gymnasiums from 2019 (Odluka o donošenju kurikuluma za nastavni predmet Matematike za osnovne škole i gimnazije u Republici Hrvatskoj (nn.hr)), learning and teaching of the subject mathematics is achieved by connecting mathematical processes and domains. This two-dimensionality manifests itself in the

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outcomes and contributes to the acquisition of mathematical competences. Mathematical processes are: Representation and Communication, Linking, Logical Reasoning, Argumentation and Inference, Problem Solving and Mathematical Modeling, and Application of Technology. The domains of mathematics subjects are: Numbers, Algebra and Functions, Shape and Space, Measurement and Data, Statistics and Probability.

According to the curriculum, despite the evolution of all concepts and processes, there is a need to change and modernize the way mathematics is learned and taught and to provide students with diverse and rich learning experiences. The ability to apply what they learn in a variety of problem situations and the knowledge to regulate their own learning are particularly important.

In organizing the learning and teaching process, the teacher selects the scope and depth of learning and adapts problems, methods, and strategies to best meet students' needs, opportunities, and interests. The teacher and students have the autonomy to select the materials and technologies that will make the learning of mathematics challenging, varied, and stimulating and will enable the realization of the intended learning outcomes. The curriculum emphasizes that the textbook in the modern mathematics classroom provides content that can be used to achieve the prescribed outcomes for all levels of knowledge, but it does not restrict the planning of the learning and teaching process or the way in which it is carried out. The teacher is free to decide how and in what order the objectives are met and what additional literature and sources of information are used by the students. The teacher is responsible for taking an innovative approach, exploring new sources of knowledge and making appropriate use of new technologies to complete the learning and teaching.

From the above it is clear that the National Curriculum does not prescribe or suggest how particular mathematical content should be taught, and significantly increases the freedom and responsibility of teachers in designing the teaching process

Enactive learning, which some of them have already used in the teaching process, has thus become an issue for a greater number of teachers, many of whom need support in taking their first steps in a new teaching approach. To help them, there is professional literature, especially texts that address the use of enactive learning in the mathematics classroom. Below are just a few of these, which are described in more detail.

3.1.1. Examples of enactive learning in Croatia

- MiŠ (Matematika i škola) – Mathematics and school (https://mis.element.hr/): Miš is an education-focused journal that is intended for school teachers, students and for anyone who is interested in mathematics. The journal is published four times a year. In the articles, various topics

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regarding the methodology of teaching mathematics are introduced. Moreover, creative and enactive activities are explained, and the latest experiences in education are presented.

Examples of enactive learning papers in MIŠ:

- Lučić, Rad s algebarskim pločicama (Working with algebra tiles), Matematika i škola XXI (2020), 105; 207-210.
- B. Majdiš, Računanje površine s pomoću tangram slagalice (Calculating area using a tangram), Matematika i škola XXI (2020), 103; 102-10.
- A. Dika, Izračunavanje površine mnogokuta s pomoću točkaste mreže (Calculating area using a dots network), Matematika i škola XX (2019), 98; 137-144. https://mis.element.hr/fajli/1709/98-11.pdf
- P. Valenčić, Matematika – nužno potrebna za život (Mathematics – necessary for life), Matematika i škola XX (2018), 97; 68-71. https://mis.element.hr/fajli/1691/97-04.pdf

- I. Brozović, S. Rukavina, Zome Tool modeli (Zome Tool models), Matematika i škola XIX (2017), 92; 51-54. https://mis.element.hr/fajli/1605/92-02.pdf
- P. Valenčić, Od ideje do izrade drvenog nastavnog pomagala (From idea to the creation of a wooden teaching aid), Matematika i škola XIX (2017), 92; 55-60. https://mis.element.hr/fajli/1606/92-03.pdf
- S. Ježić, Božićna zvijezda – vertikalno povezivanje (Christmas star – vertical connecting), Matematika i škola XIX (2017), 92; 68-72. https://mis.element.hr/fajli/1609/92-06.pdf
- T. Sabo, S. Rukavina, Origami i krivulje drugog reda (Origami and second-order curves), Matematika i škola XVIII (2016), 86; 20-22. https://mis.element.hr/fajli/1486/86-05.pdf
- M. Černivec, S. Rukavina, Radionica "Kombinatorne igre" (Workshop "Combinatorial games"), Matematika i škola XIII (2011), 61; 14-16. https://mis.element.hr/fajli/1087/61-04.pdf

- Poučak (https://matematika.hr/izdanja/poucak/): Poučak is an education-focused journal for methodology of teaching mathematics. It was founded by The Croatian Mathematical Society which promotes mathematical science, teaching mathematics at all levels, applying mathematics in other disciplines as well as improving the social position of mathematicians as a whole. The journal is published four times a year.

- Matka (https://matematika.hr/izdanja/matka/): Matka is a journal intended for students in primary school and in lower grades of secondary school, for their teachers, and parents as well. Various topics from geometry, arithmetic, algebra and history of mathematics are included. Moreover, an application of mathematics in other sciences and art, problems for talented students, math

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games and rebus puzzles can be also found within the journal. The journal is published four times a year.

- Acta Mathematica Spalatensia (https://amas.pmfst.unist.hr/ams/): is an international journal devoted to publication of articles in all areas of pure and applied mathematics. The journal publishes original research papers and high-quality review articles.

  The Development of Scientific and Mathematical Literacy Using Active Learning Strategies:

- S. Rukavina, B. Milotić, R. Jurdana-Šepić, M. Žuvić-Butorac, J. Ledić, Razvoj prirodoznanstvene i matematičke pismenosti aktivnim učenjem (The Development of Scientific and Mathematical Literacy Using Active Learning Strategies), Udruga Zlatni rez, 2010.
- The book “The Development of Scientific and Mathematical Literacy Using Active Learning Strategies” describes the teaching challenges for the natural sciences and mathematics and points out the importance of enactive learning. The book includes 12 workshops - 6 workshops related to mathematics and 6 workshops related to physics.

3.2. German philosophy

The national standards define for students’ processes and activities that, is the orientation for learning mathematics. The national standards for mathematical learning based on three basic experiences by Winter (1996). These experiences can be characterized as (E1) application orientation, (E2) structure orientation and (E3) problem orientation. Thereby, application orientation (E1) does not directly mean the preparation for specific situations in life, but rather, the possibility of a basic insight into nature, society and culture. Structure orientation (E2) looks more at the analysis of mathematical objects in relation to a deductive view of the world. The problem orientation (E3) on the other hand, emphasizes the acquisition of heuristic abilities to recognize and use samples in problem solving processes. However, these three aspects are connected with each other.

The learning environment in mathematics at school should enable students to make these experiences. Enactive learning can be seen as part of E1 and E3, where students get in contact with mathematics as part of real-life situation and modeling processes. To enable the basic experiences, competencies are defined as an aim of mathematical learning. The process oriented competencies describe central activities and processes for the development of mathematics understanding: mathematical argumentation (K1), problem solving (K2), mathematical modeling (K3), use of mathematical representations (K4), use of symbolic, formal and technical elements of mathematics (K5), and mathematical communication (K6).

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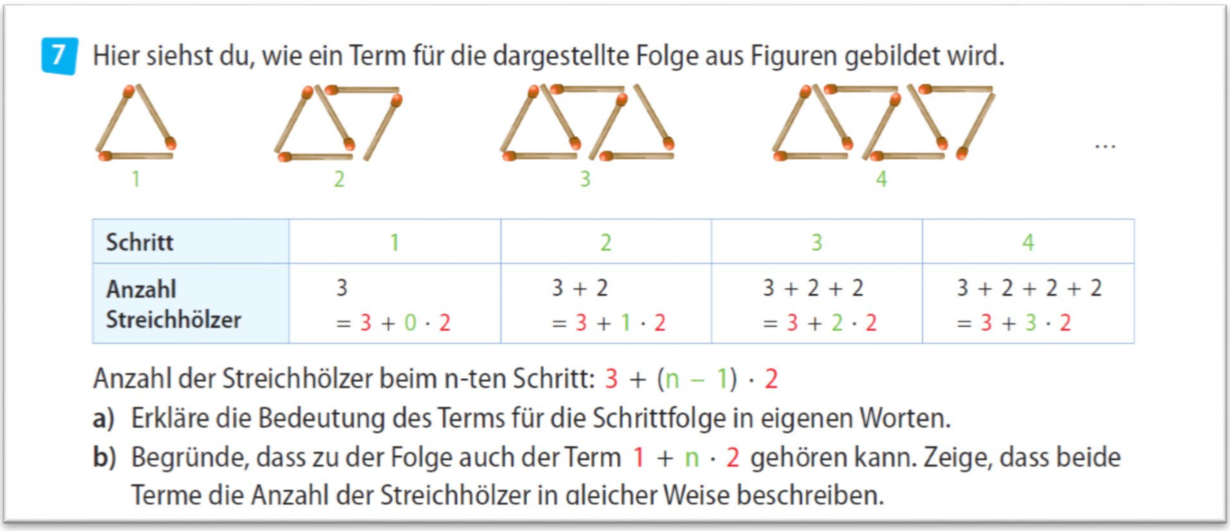
3.2.1. Examples of enactive learning in Germany



7 Hier siehst du, wie ein Term für die dargestellte Folge aus Figuren gebildet wird.

Schritt	1	2	3	4
Anzahl Streichhölzer	3 = 3 + 0 · 2	3 + 2 = 3 + 1 · 2	3 + 2 + 2 = 3 + 2 · 2	3 + 2 + 2 + 2 = 3 + 3 · 2

Anzahl der Streichhölzer beim n-ten Schritt: $3 + (n - 1) \cdot 2$

a) Erkläre die Bedeutung des Terms für die Schrittfolge in eigenen Worten.

b) Begründe, dass zu der Folge auch der Term $1 + n \cdot 2$ gehören kann. Zeige, dass beide Terme die Anzahl der Streichhölzer in gleicher Weise beschreiben.

Figure 6: Enactive learning in Textbooks.

In the field of enactive learning there the following aspects of competencies are described in the national standards and curricula:

- *mathematical argumentation (K1): This competence includes the understanding and evaluation of mathematical proofs, the development of sequences of mathematical argumentation and assumptions.* For enactive learning this competence is deeply connected with the E-I-S principle of Bruner by handling with concrete material for mathematical assumption and argumentation.
- *problem solving (K2): This competence includes the use of strategies and heuristics for solving mathematical problems and their applications. This can include known strategies and the try and connection of new strategies. Solution plans and results are proofed and critically reflected.*

Figure 7: Enactive example of exponential function

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This competence describes a meta-level of mathematical working. With regard to enactive learning also enactive strategies for systematic proving of mathematical problems can be found.

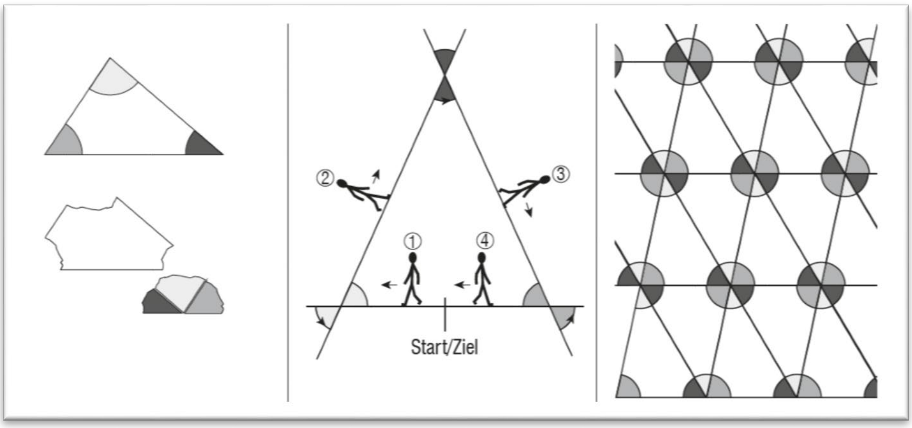

Figure 8: Enactive example Sum of angles in a triangle.

- *mathematical modeling (K3): This competence includes the translation between mathematics and real world situations, also the concepts, methods, and models.*
  For enactive learning, this translation between real world and mathematical world is one central aspect for describing, finding, and using mathematical concepts and models for understanding a given enactive situation and task.
- *use of mathematical representations (K4): This competence includes the representation, their selection, differentiation, and making of representation types.*
  For enactive learning the E-I-S Principle of Bruner is deeply connected to this competence.
- *use of symbolic, formal and technical elements of mathematics (K5): this competence includes the facts and mathematical rules, the use of algorithms for the algebraic and geometric operations. Thereby the senseful and reflected use of mathematical tools is part of this competence.*
  For enactive learning the solving of enactive tasks, the use of mathematical rules is in mind.
- *mathematical communication (K6): This competence includes the understanding of mathematical given verbal and written information. Moreover the documentation of solution plans, the use of mathematical concepts with respect to different audiences are also part of this competence.*
  For enactive learning this competence takes the communication process in mind: from the understanding of the given tasks or situation up to the documentation (e.g. the use or making of mathematical videos).

3.3. Lithuanian philosophy

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According to the project on math program (2021), the subject of math at school plays a unique role in the development of student's skills of numeracy, abstract, logical thinking, visual, spatial thinking, data analysis and interpretation, abstraction skills.

The objectives of secondary education in math are as follows, students:

- use mathematical concepts properly and purposefully, indicate and explain the connections between them;
- perform mathematical procedures smoothly;
- recognize and apply mathematical reasoning in various contexts;
- organize his / her learning activities responsibly and effectively;
- communicate effectively through mathematical language;
- see the connections between math and other subjects;
- use digital technologies to learn mathematics;
- are confident, collaborative, think critically and effectively adapt acquired knowledge and skills in math to the solution of problems they understand in a variety of contexts.

In all grades, from first to twelfth, student achievement is projected in three areas of achievement: deep understanding and reasoning, mathematical communication and problem solving.

However, although math national program emphasizes the importance of the development of creative thinking competence, the most of math lessons are of the standard type: tasks and assignments from textbooks/handbooks, solutions are explained by the teacher, for example, an excerpt from the textbook for 9th grade:

Figure 9: Examples of creative thinking

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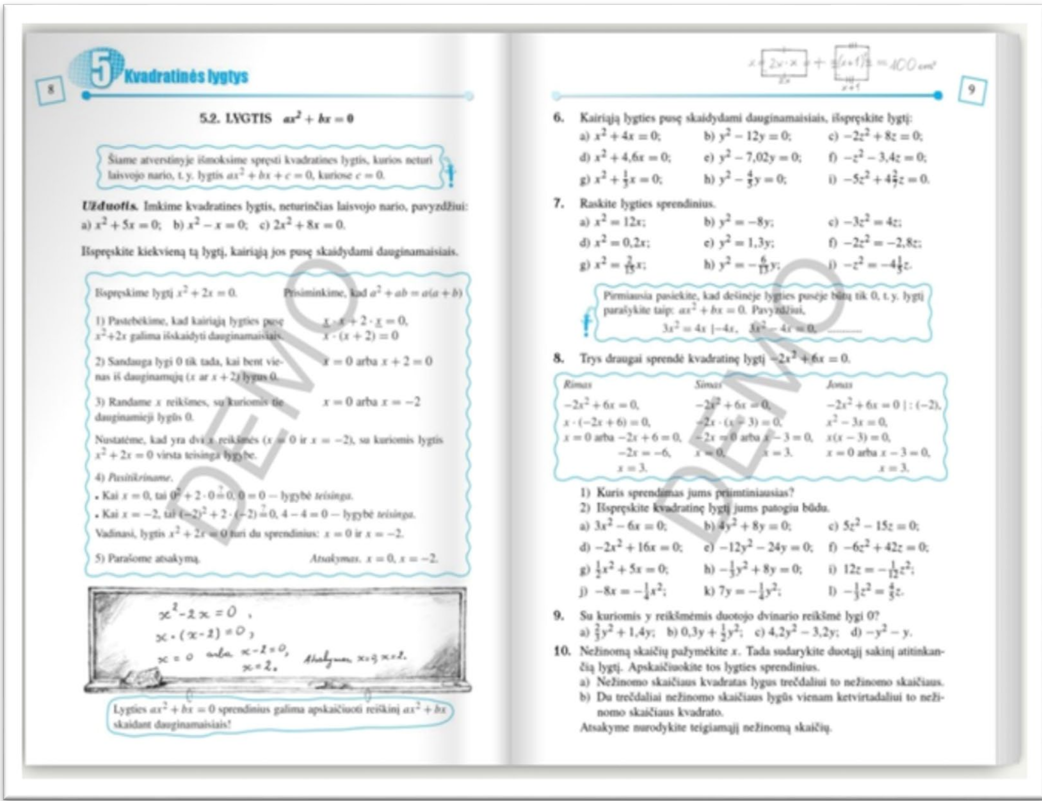

8

5 Kvadratinės lygtys

5.2. LYGTIS $ax^2 + bx = 0$

Šiame atverstinyje išmoksime spręsti kvadratines lygtis, kurios neturi laisvojo nario, t. y. lygtis $ax^2 + bx + c = 0$, kuriose $c = 0$.

Užduotis. Imkime kvadratines lygtis, neturinčias laisvojo nario, pavyzdžiui: a) $x^2 + 5x = 0$; b) $x^2 - x = 0$; c) $2x^2 + 8x = 0$.

Išspręskite kiekvieną tą lygtį, kairiąją jos pusę skaidydami dauginamaisiais.

Išspręskime lygtį $x^2 + 2x = 0$. Prisiminkime, kad $a^2 + ab = a(a + b)$

1) Pastebėkime, kad kairiąją lygties pusę x^2+2x galima išskaidyti dauginamaisiais. $x \cdot x + 2 \cdot x = 0$, $x \cdot (x + 2) = 0$

2) Sandauga lygi 0 tik tada, kai bent vienas iš dauginamųjų (x ar $x + 2$) lygus 0. $x = 0$ arba $x + 2 = 0$

3) Randame x reikšmes, su kuriomis tie dauginamieji lygūs 0. $x = 0$ arba $x = -2$

Nustatėme, kad yra dvi x reikšmės ($x = 0$ ir $x = -2$), su kuriomis lygtis $x^2 + 2x = 0$ virsta teisinga lygybe.

4) *Pasitikriname.*

- Kai $x = 0$, tai $0^2 + 2 \cdot 0 \overset{?}{=} 0$, $0 = 0$ – lygybė *teisinga*.
- Kai $x = -2$, tai $(-2)^2 + 2 \cdot (-2) \overset{?}{=} 0$, $4 - 4 = 0$ – lygybė *teisinga*.

Vadinasi, lygtis $x^2 + 2x = 0$ turi du sprendinius: $x = 0$ ir $x = -2$.

5) Parašome atsakymą. *Atsakymas.* $x = 0$, $x = -2$.

$x^2 - 2x = 0$,
$x \cdot (x - 2) = 0$,
$x = 0$ arba $x - 2 = 0$,
$x = 2$. Atsakymas. $x = 0$; $x = 2$.

Lygties $ax^2 + bx = 0$ sprendinius galima apskaičiuoti reiškinį $ax^2 + bx$ skaidant dauginamaisiais!

9

6. Kairiąją lygties pusę skaidydami dauginamaisiais, išspręskite lygtį:
a) $x^2 + 4x = 0$; b) $y^2 - 12y = 0$; c) $-2z^2 + 8z = 0$;
d) $x^2 + 4{,}6x = 0$; e) $y^2 - 7{,}02y = 0$; f) $-z^2 - 3{,}4z = 0$;
g) $x^2 + \frac{1}{3}x = 0$; h) $y^2 - \frac{4}{5}y = 0$; i) $-5z^2 + 4\frac{2}{7}z = 0$.

7. Raskite lygties sprendinius.
a) $x^2 = 12x$; b) $y^2 = -8y$; c) $-3z^2 = 4z$;
d) $x^2 = 0{,}2x$; e) $y^2 = 1{,}3y$; f) $-2z^2 = -2{,}8z$;
g) $x^2 = \frac{2}{15}x$; h) $y^2 = -\frac{6}{13}y$; i) $-z^2 = -4\frac{1}{5}z$.

Pirmiausia pasiekite, kad dešinėje lygties pusėje būtų tik 0, t. y. lygtį parašykite taip: $ax^2 + bx = 0$. Pavyzdžiui,
$3x^2 = 4x \mid -4x$, $3x^2 - 4x = 0$.

8. Trys draugai sprendė kvadratinę lygtį $-2x^2 + 6x = 0$.

Rimas	*Simas*	*Jonas*
$-2x^2 + 6x = 0$,	$-2x^2 + 6x = 0$,	$-2x^2 + 6x = 0 \mid : (-2)$,
$x \cdot (-2x + 6) = 0$,	$-2x \cdot (x - 3) = 0$,	$x^2 - 3x = 0$,
$x = 0$ arba $-2x + 6 = 0$,	$-2x = 0$ arba $x - 3 = 0$,	$x(x - 3) = 0$,
$-2x = -6$,	$x = 0$, $x = 3$.	$x = 0$ arba $x - 3 = 0$,
$x = 3$.		$x = 3$.

1) Kuris sprendimas jums priimtiniausias?
2) Išspręskite kvadratinę lygtį jums patogiu būdu.
a) $3x^2 - 6x = 0$; b) $4y^2 + 8y = 0$; c) $5z^2 - 15z = 0$;
d) $-2x^2 + 16x = 0$; e) $-12y^2 - 24y = 0$; f) $-6z^2 + 42z = 0$;
g) $\frac{1}{2}x^2 + 5x = 0$; h) $-\frac{1}{3}y^2 + 8y = 0$; i) $12z = -\frac{1}{12}z^2$;
j) $-8x = -\frac{1}{4}x^2$; k) $7y = -\frac{1}{4}y^2$; l) $-\frac{1}{3}z^2 = \frac{4}{5}z$.

9. Su kuriomis y reikšmėmis duotojo dvinario reikšmė lygi 0?
a) $\frac{2}{3}y^2 + 1{,}4y$; b) $0{,}3y + \frac{1}{2}y^2$; c) $4{,}2y^2 - 3{,}2y$; d) $-y^2 - y$.

10. Nežinomą skaičių pažymėkite x. Tada sudarykite duotąjį sakinį atitinkančią lygtį. Apskaičiuokite tos lygties sprendinius.
a) Nežinomo skaičiaus kvadratas lygus trečdaliui to nežinomo skaičiaus.
b) Du trečdaliai nežinomo skaičiaus lygūs vienam ketvirtadaliui to nežinomo skaičiaus kvadrato.
Atsakyme nurodykite teigiamąjį nežinomą skaičių.

Although, the math tasks can have practical aspects, as in the example of the math task for 8th grade students. It was told how the Koch snowflake is made (but it was not asked to make it by themselves) and then it was asked to calculate various parameters. Some other tasks may ask to find a solution for a particular life situation, mentioned in the task, for example, to calculate the distance etc.

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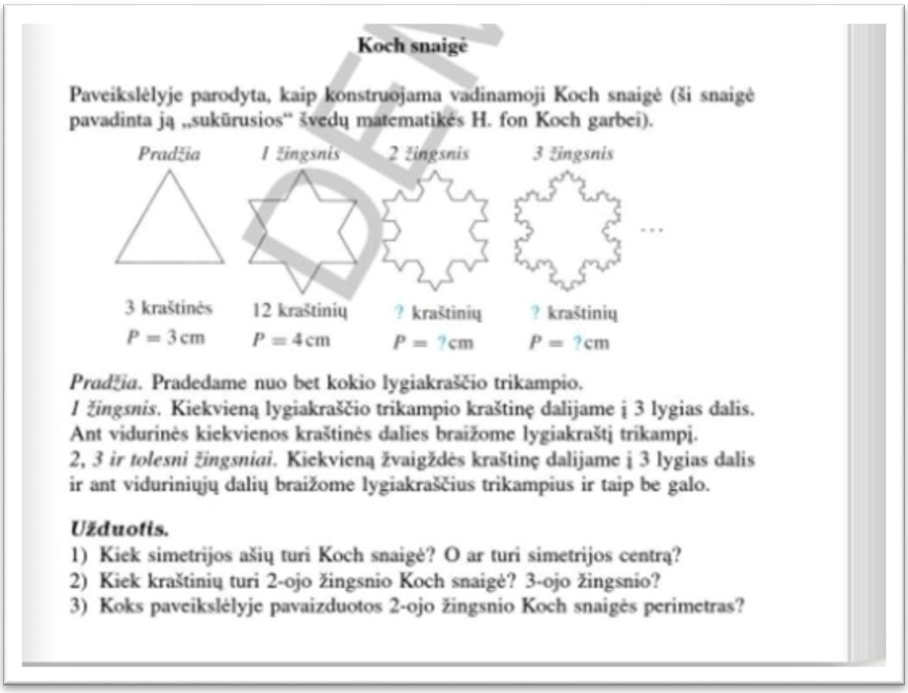

Koch snaigė

Paveikslėlyje parodyta, kaip konstruojama vadinamoji Koch snaigė (ši snaigė pavadinta ją „sukūrusios" švedų matematikės H. fon Koch garbei).

Pradžia. Pradedame nuo bet kokio lygiakraščio trikampio.
1 žingsnis. Kiekvieną lygiakraščio trikampio kraštinę dalijame į 3 lygias dalis. Ant vidurinės kiekvienos kraštinės dalies braižome lygiakraštį trikampį.
2, 3 ir tolesni žingsniai. Kiekvieną žvaigždės kraštinę dalijame į 3 lygias dalis ir ant viduriniųjų dalių braižome lygiakraščius trikampius ir taip be galo.

Užduotis.
1) Kiek simetrijos ašių turi Koch snaigė? O ar turi simetrijos centrą?
2) Kiek kraštinių turi 2-ojo žingsnio Koch snaigė? 3-ojo žingsnio?
3) Koks paveikslėlyje pavaizduotos 2-ojo žingsnio Koch snaigės perimetras?

Figure 10: Example of active learning methods

There are also active learning methods: group work, project research, competitions, discussions, adaptation of IT tools (creating mind maps, quizzes, logic games, crosswords, using 3D figures, etc.). However, since math exams are in a written form with a lot of calculation, the main task of math teachers is to give the students the understanding and skills of how to calculate quickly and thus solve as many tasks as possible. More enactive tasks are given to primary school students because there is more time and freedom to work on creative tasks – to draw, to make various figures, etc.

The importance and ways to make math teaching and learning more involving and creative, are analyzed for decades in scientific literature, for example, how to draw a cat using the lines and functions (for the 9th grade students). Biekšienė, R., ir Zenkevičienė M. (2000). Aktyvaus mokymosi metodai matematikos pamokose. α+ώ, Nr. 2, 52-56 (Figure 11):

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I koordinatiniame ketvirtyje:

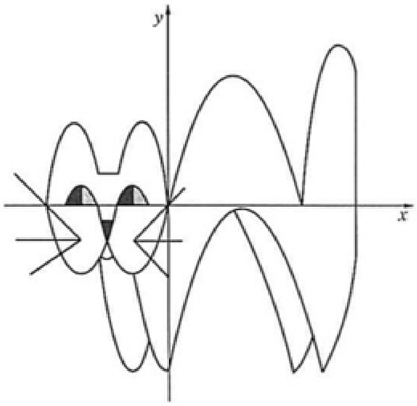
1) $y=-\frac{1}{2}(x-4)^2+8$, kai $0 \leqslant x \leqslant 8$;
2) $y=-2(x-10)^2+10$, kai $7{,}8 \leqslant x \leqslant 11$;
3) $x=11$, kai $0 \leqslant y \leqslant 8{,}2$;
4) $y=x$, kai $0 \leqslant x \leqslant 1$;
5) $y=-2(x+1{,}5)^2+5{,}2$, kai $0 \leqslant x \leqslant 0{,}2$.

II koordinatiniame ketvirtyje:

1) $y=-x-7$, kai $-9 \leqslant x \leqslant -7$;
2) $y=-2(x+1{,}5)^2+5{,}2$, kai $-2{,}8 \leqslant x \leqslant 0$;
3) $y=-2(x+5{,}5)^2+5{,}2$, kai $-7{,}2 \leqslant x \leqslant -4$;
4) $y=2$, kai $-4 \leqslant x \leqslant -2{,}8$;
5) $y=-2(x+2)^2+1{,}2$, kai $-2{,}8 \leqslant x \leqslant -1{,}2$;
6) $y=-2(x+5)^2+1{,}2$, kai $-5{,}8 \leqslant x \leqslant -4{,}2$;
7) $y=0$, kai $-5{,}8 \leqslant x \leqslant -4{,}2$;
8) $y=-5$, kai $0 \leqslant x \leqslant 1{,}2$ (nuspalvinkite dešiniąją akies pusę pilkai, o kairiąją — juodai);
9) $y=0$, kai $-2{,}8 \leqslant x \leqslant -1{,}2$;
10) $x=-2$, kai $0 \leqslant x \leqslant 1{,}2$ (nuspalvinkite dešiniąją akies pusę pilkai, o kairiąją — juodai).

III koordinatiniame ketvirtyje:

1) $y=(x+2)^2-4$, kai $-4 \leqslant x \leqslant 0$;
2) $y=(x+5)^2-4$, kai $-7{,}2 \leqslant x \leqslant -3$;
3) $y=-1$, kai $-3{,}8 \leqslant x \leqslant -3{,}2$ (nuspalvinkite nosį juodai);
4) $y=2(x+3{,}5)^2-3$, kai $-3{,}8 \leqslant x \leqslant -3{,}2$ (nuspalvinkite liežuvį pilkai);
5) $y=2x^2-10$, kai $-1{,}8 \leqslant x \leqslant 0$;
6) $y=2(x+2)^2-10$, kai $-4 \leqslant x \leqslant -1$;
7) $y=-2$, kai $-9 \leqslant x \leqslant -5$, $-2 \leqslant x \leqslant 0$;



8) $y=x$, kai $-2 \leqslant x \leqslant 0$;
9) $y=-x-4$, kai $-2 \leqslant x \leqslant 0$;
10) $y=-x-7$, kai $-7 \leqslant x \leqslant -5$;
11) $y=\frac{2}{3}x+\frac{4}{3}$, kai $-8 \leqslant x \leqslant -5$.

IV koordinatiniame ketvirtyje:

1) $y=-\frac{1}{2}(x-4{,}5)^2$, kai $0 \leqslant x \leqslant 9$;
2) $y=-\frac{1}{2}(x+2{,}5)^2+1$, kai $4 \leqslant x \leqslant 7{,}4$;
3) $y=2(x-9)^2-10$, kai $9 \leqslant x \leqslant 11$;
4) $x=11$, kai $-3 \leqslant y \leqslant 0$;
5) $y=2(x-7{,}4)^2-10$, kai $7{,}4 \leqslant x \leqslant 8{,}5$;
6) $y=-2$, kai $0 \leqslant x \leqslant 1$.

Figure 12: Example of creative thinking wit graphs

In 2020, only 67.61% of candidates in Lithuania passed the math exam (in comparison to 2019 - 82.09% of candidates).It reveals a significant importance to adapt a more enactive learning approach to gain a better understanding in math because failed high school students emphasize that they were always bad in math lessons and don’t think that they need much math in their life later. In addition, math teachers should get a more systematic approach regarding enactive teaching of math in order to help the students to understand the importance of math in their life and train their competences in math based problem solving.

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3.4. Spanish philosophy

In tendency, schools in Spain, whether private or public, have a similar model of teaching in which enactive learning has no place. After an exhaustive search for schools that teach with enactive approach, we have found hardly any institutions that use it in their process of teaching. In this sense, the approach that most resembles enactive learning is the Montessori Method, which is increasingly being used in Spain.

There has been a change in pedagogy in many institutes and schools, as it is increasingly common to find alternative schools with more person-center teaching and learning about emotions and development of other intelligences, broadening the focus beyond the theoretical knowledge that has always existed in school. In this way and following the above mentioned about the lack of use of the enactive approach in Spanish schools but the increasing openness in pedagogy, EnLeMaH project makes sense in Spain. It has a place for teachers, who are constantly trying to open up new ways of looking at and approaching students, so this Project will give them new tools that have been proven in other countries and are successful.

Some Spanish schools and institutes are beginning to innovate with project-based learning, in which transversal competencies are increasingly valued as well as the creation and work in groups of the content to be learned. Some of the highlight schools and institutes in Spain because of their innovations are:

- Escuela Ideo, Madrid: they work through the project-based learning model. Group work for learning and learning by doing are two of their core.
- Fundación Myland, Seville: this school is currently constructing the building that will house the secondary school. Their pedagogy is based on experiential learning, which makes it easier for students to acquire knowledge through learning by doing.
- Colegio San Gregorio, Plasencia: Subsidized educational center with pupils from 0 to 18 years of age. The aim of its teaching staff is to structure learning through projects and active methodologies. Emotional education is encouraged and in primary education the Heroes Tic methodology is used, which makes pupils the protagonists of their learning experience. The keynote is that cooperative groups are used and the children's progress is recorded in the pupils' digital blogs.
- Colegio Amara Berri: The center’s educational methodology has age-based cycle programs and special care is taken to teach pupils ways of organizing their thinking and methods for its development, as well as working on pupils' self-esteem and promoting teamwork. Gamification and the practical application of theoretical knowledge also play an important role.

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There are many Montessori schools in Spain but teaching is usually up to the age of 12, so although the method is very similar to Enactive Learning, the age range is not the same.

In conclusion, therefore, the EnLeMaH project will have a very significant and necessary impact on Spanish pedagogy as it will give teachers numerous tools and tips to be able to adapt their content to the learning needs of their students, making their teaching much more meaningful than it has been so far.

3.5. Conclusion

As mentioned in the different country-specific philosophy there can be observed differences in the way, how to founded enactive learning at school. The differences can be more in the way of implementation enactive learning than in the way of underline the necessity of enactive learning at school. In all countries the process of integrating teachers in this philosophy as an important question for establish enactive learning at school. In this field the EnLeMaH-project is part of the strategy and the philosophy of each country.

4. EnLeMaH and the criteria for an enactive work.

On the base of chapter 1 and 3, the following criteria for enactive learning in the EnLeMaH-project will be the basis for the creation of enactive learning situations. Thereby the aspects of the enactive representation mode, the experimental point of view and the philosophy of enactive learning will be bundled in the following criteria:

- Real object: Learning environments must contain real objects. (Computer-based activities (such as GeoGebra) is not considered an enactive activity).
- Activity: Action is an active process. Learners should not be passive recipients but being actively involved in the learning process (e.g., watching an experiment but not conducting it themselves). (Learners must be involved in activities.) Students must be active in every step of an activity.
- Lesson: Enactive activities can be part of every lesson element (e.g., introduction, learning new content, automatizing, concluding...)
- Material: Tangible things must be accessible by students at home (or in class)
- Setting of enactive activities at home: Synchronous (live) and asynchronous (self-learning without live presentation). The setting of enactive learning at home has next to tangible material the opportunity to enable teachers to create different kind of distance learning environments.

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5. The template for the EnLeMaH

The following template should be the basis of the documentation of a created enactive work. Thereby aspects of practical teaching and an easy use of this activity for teachers from other countries or outside the EnLeMaH-project will take into account. The documentation enables teachers to understand and to adapt activities to their own classes.

Table 7: Template for the EnleMaH

Name of the activity
Abstract (mother language)
Abstract (English)
Purpose of the task (introduction, exercise, repetition...)
Learning outcomes
Students' prior knowledge
List of material
Expected time for the activity
Expected time for the preparation

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Short description of the activity (the description should be divided into few parts, according to the tasks. One should predict the duration of each task)
Exercises for students (it should be written in a separate document)
Solution plan (it should be written in a separate document)
Remarks (hints, difficulties, classroom management, differentiation, possibilities to extend the activity)

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