The dynamic behaviour of an endogenous growth model with public capital and pollution

Alfred Greiner*

Abstract

In this paper we present an endogenous growth model with environmental pollution and public capital. As to pollution we assume that it is a by-product of aggregate production and that it negatively affects utility of the household but not production possibilities directly. The paper studies the dynamics of the model and demonstrates that there exists either a unique balanced growth path which is a saddle point or there exist two balanced growth paths with one being locally saddle point stable and one being asymptotically stable.

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*Department of Business Administration and Economics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany, e-mail: agreiner@wiwi.uni-bielefeld.de

1. Introduction

In endogenous growth theory models exist which are characterized by local and/or global indeterminacy. Local indeterminacy can explain why some countries may have different growth rates along the transition path but the same growth rate in the long-run. Global indeterminacy means that there are different long-run growth paths to which economies converge, implying that not only the transitional growth rates differ but also the long-run growth rates.

One economic prerequisite for indeterminacy are external effects. For example, Benhabib and Farmer (1994) demonstrate that an endogenous growth model with positive externalities (of capital and of labour) and with elastic labour supply can generate both local and global indeterminacy. The economic condition for indeterminacy to occur in this model is that the slope of the labour demand curve exceeds the slope of the supply curve, besides the positive externalities. Benhabib and Perli (1994) show that the human capital model of endogenous growth can also lead to indeterminacy even for an exogenously given labour supply. The economic mechanisms leading to indeterminate equilibrium paths are a sufficiently strong externality of human capital in the production process of the physical good and a sufficiently high intertemporal elasticity of substitution of consumption.

The latter also holds in the model presented by Greiner (2003). There, it is demonstrated that an endogenous growth model with positive externalities of investment can lead to indeterminate equilibrium paths if the elasticity of substitution is sufficiently high. The difference to the contribution by Benhabib and Farmer (1994) is that the physical capital stock and the knowledge stock are treated as two different state variables and that labour is exogenous.

While those studies analyze the role of externalities which positively affect the supply side of the economy, we intend to study in this paper effects of negative externalities on the utility of the household as concerns its influence on the dynamics. Such a negative externality is environmental pollution.

We focus on environmental pollution and its effect on the dynamics of an endogenous growth model because most papers dealing with the environment are interested in growth and welfare effects of fiscal policy (see for example Bovenberg and Smulders, 1995, Gradus and Smulders, 1993, Bovenberg and de Mooij, 1997, or Hettich, 1998, just to mention a few). But those models do not have transition dynamics or the analysis is limited to the balanced growth path. An explicit analysis of the dynamics is most often beyond the scope of these contributions. An exception is provided by the paper by Koskela et al. (2000) who study an overlapping generations model with a renewable resource which serves as a store of value and as an input factor in the production of the consumption good. They find that indeterminacy and cycles may result in their model.

In this paper we want to analyze the role of environmental pollution as to the emergence of indeterminacy in a competitive economy of an endogenous growth model, in particular we want to find which mechanisms can generate this phenomenon. To do so, we present a model of economic growth with productive public capital, as in Barro (1990) and Futagami et al. (1993), which is the source of sustained growth. In addition, we assume that there are negative externalities of production in the form of environmental pollution, which reduce utility of the household.

It also should be mentioned integrating pollution, which is a function of output and, thus, of capital, is likely to generate indeterminacy. This holds because Kurz (1968) and Levhari and Liviatan (1973) have demonstrated that the inclusion of capital in the utility function affects the number of steady states and stability, respectively, in exogenous growth models. But it must be underlined that their result is derived for exogenous growth models and cannot be applied to endogenous growth models one-to-one.

The rest of the paper is organized as follows. In section 2 we present the structure of our model. Section 3 studies the dynamics of the model and in section 4 we summarize and conclude the paper.

2. Structure of the model

We consider a decentralized economy which comprises three sectors: the household sector, a productive sector, and the government.

2.1 The household

Our economy consists of one representative household which maximizes its discounted stream of utility subject to its budget constraint:

$$J(\cdot) \equiv \max_{C(t)} \int_0^\infty e^{-\rho t} V(t) dt, \tag{2.1}$$

with $V(t)$ the instantaneous subutility function which depends positively on the level of consumption, $C(t)$, and negatively on effective pollution, $P_E(t)$. $V(t)$ takes the following form

$$V(t) = (C(t) P_E(t)^{-\xi})^{1-\sigma} / (1-\sigma), \tag{2.2}$$

where $\xi > 0$ gives the disutility arising from effective pollution.[1] $1/\sigma > 0$ gives the intertemporal elasticity of substitution of private consumption between two points in time for a given level of effective pollution and $\ln$ is the natural logarithm. ρ in (2.1) is the subjective discount rate and there is no population growth.

[1]For a survey of how to incorporate pollution in the utility function see Smulders (1995), p. 328-29.

The budget constraint is given by[2]

$$\dot{K} = -C + wL + rK, \tag{2.3}$$

with L denoting labour which is supplied inelastically. The budget constraint (2.3) states that the individual has, as usual, to decide how much to consume and how much to save, thus increasing consumption possibilities in the future. The depreciation of physical capital is assumed to equal zero. w in the budget constraint is the wage rate and r is the return to capital K.

Assuming that a solution to (2.1) subject to (2.3) exists we can use the current-value Hamiltonian to describe that solution. The Hamiltonian function is written as

$$\mathcal{H}(\cdot) = (CP_E^{-\xi})^{1-\sigma}/(1-\sigma) + \lambda(-C + wL + rK),$$

with λ the costate variable. The necessary optimality conditions are given by

$$\begin{aligned} \lambda &= C^{-\sigma} P_E^{-\xi(1-\sigma)}, && (2.4) \\ \dot{\lambda}/\lambda &= \rho - r, && (2.5) \\ \dot{K} &= -C + wL + rK. && (2.6) \end{aligned}$$

Since the Hamiltonian is concave in C and K jointly, the necessary conditions are also sufficient if in addition the transversality condition at infinity $\lim_{t\to\infty} e^{-\rho t}\lambda(t)(K(t) - K^\star(t)) \geq 0$ is fulfilled with $K^\star(t)$ denoting the optimal value. Moreover, strict concavity in C also guarantees that the solution is unique (cf. Seierstad and Sydsaeter (1987), pp. 234-235).

2.2 The productive sector

The productive sector in our economy can be represented by one firm which chooses inputs in order to maximize profits and which behaves competitively. As to pollution, we suppose that it is the result of aggregate production. In particular, we assume that pollution $P(t)$ is a by-product of output $Y(t)$, i.e. $P(t) = \varphi Y(t)$, with $\varphi = const. > 0$. Thus, we follow the line invited by Forster (1973) and worked out in more details by Luptacik and Schubert (1982).

The production function is given by,

$$Y = K^\alpha L^{1-\alpha} H^{1-\alpha}, \tag{2.7}$$

with H denoting the stock of productive public capital which raises efficiency of labour and $\alpha \in (0,1)$ gives the capital share. Pollution is taxed at the rate $\tau_p > 0$ and the firms take into account that one unit of output causes φ units of pollution for which they have to pay $\tau_p\varphi < 1$ per unit of output.

[2] In what follows we will suppress the time argument if no ambiguity arises.

The optimization problem of the firm then is given by

$$\max_{K,L} K^{\alpha}L^{1-\alpha}H^{1-\alpha}(1-\varphi\tau_p) - rK - wL \tag{2.8}$$

Assuming competitive markets and taking public capital as given optimality conditions for a profit maximum are obtained as

$$\begin{aligned} w &= (1-\tau_p\varphi)(1-\alpha)L^{-\alpha}K^{\alpha}H^{1-\alpha}, && (2.9)\\ r &= (1-\tau_p\varphi)\alpha K^{\alpha-1}H^{1-\alpha}L^{1-\alpha}. && (2.10) \end{aligned}$$

2.3 The government

The government in our economy receives tax revenue from the taxation of pollution. The tax revenue is spent for abatement activities $A(t)$ which reduce total pollution and for the formation of public capital, $H(t)$. Abatement activities are determined by $A(t) = \eta\tau_p P(t)$, with $\eta < 1$. $\eta < 1$ means that not all of the pollution tax revenue is used for abatement activities and the remaining part is used for public investment in the public capital stock I_p, $I_p > 0$.

However, pollution cannot be eliminated completely. We call that part of pollution which remains in spite of abatement activites the effective pollution $P_E(t)$. In particular, we follow Gradus and Smulders (1993) and Lighthart and van der Ploeg (1994) and take the following specification

$$P_E = \frac{P}{A^{\beta}}\,,\ 0 < \beta \leq 1. \tag{2.11}$$

The limitation $\beta \leq 1$ assures that a positive growth rate of aggregate production goes along with an increase in effective pollution, $\beta < 1$, or leaves effective pollution unchanged, $\beta = 1$. This implies that effective pollution cannot decline as economies grow over time. A justification for this may be seen in irreversibility of environmental damages. But it must be stated that our formulation can only be seen as a crude approximation to the irreversibility problem because a proper formulation would require modelling the environment as a stock which, however, is beyond the scope of this paper.

Moreover, the government in our economy runs a balanced budget at any moment in time. Thus, the budget constraint of the government is written as

$$I_p + A = \tau_p P \leftrightarrow I_p = \tau_p P(1-\eta). \tag{2.12}$$

The evolution of public capital is described by

$$\dot{H} = I_p, \tag{2.13}$$

where for simplicity we again neglect depreciation of public capital.

3. The dynamics of the model

In the following, labour is normalized to one, i.e. $L \equiv 1$. An equilibrium allocation in the economy, then, is given if K and L maximize profits of the firm, C maximizes (2.1) and the budget of the government is balanced.

Profit maximization of the firm implies that the marginal products of capital and of labour equal the interest rate and the wage rate. This implies that in equilibrium the growth rate of physical capital is given by

$$\frac{\dot{K}}{K} = -\frac{C}{K} + \left(\frac{H}{K}\right)^{1-\alpha}(1-\varphi\tau_p), \; K(0) = K_0. \tag{3.14}$$

Using the budget constraint of the government the growth rate of public capital is

$$\frac{\dot{H}}{H} = \left(\frac{H}{K}\right)^{-\alpha}\varphi\tau_p(1-\eta), \; H(0) = H_0. \tag{3.15}$$

Utility maximization of the household yields the growth rate of consumption as

$$\frac{\dot{C}}{C} = -\frac{\rho}{\sigma} + \sigma^{-1}(1-\varphi\tau_p)\alpha\left(\frac{H}{K}\right)^{1-\alpha} - \xi(1-\beta)\frac{1-\sigma}{\sigma}\left(\alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{H}}{H}\right). \tag{3.16}$$

Equations (3.14), (3.15) and (3.16) completely describe the economy in equilibrium. The initial conditions $K(0) = K_0$ and $H(0) = H_0$ are given and fixed and $C(0)$ can be chosen freely by the economy. Further, the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda(t)(K(t) - K^\star(t)) \geq 0$ must be fulfilled, with $K^\star(t)$ denoting the optimal value and λ determined by (2.4).[3]

Before we study the dynamics of our model we have to define a balanced growth growth (BGP). This is done in the following theorem.

Definition 1 *A balanced growth path (BGP) is path on which the ratios $c \equiv C/K$ and $h \equiv H/K$ are constant.*

This definition implies that C, H, K and Y grow at the same rate on a BGP. As concerns pollution, it is either constant on the BGP, if $\beta = 1$, or it grows but at a smaller rate, if $\beta < 1$. Thus, we follow that line of research which allows for rising pollution as economies grow as e.g. Lighthart and van der Ploeg (1994) or Byrne (1997).

With definition 1, the system describing the dynamics around a BGP is written as

$$\begin{aligned}\frac{\dot{c}}{c} &= -\frac{\rho}{\sigma} + \frac{\alpha h^{1-\alpha}(1-\varphi\tau_p)}{\sigma} - (1-\alpha)\xi(1-\beta)\frac{1-\sigma}{\sigma}h^{-\alpha}\varphi\tau_p(1-\eta) + \\ &\quad \left(1 + \alpha\xi(1-\beta)\frac{1-\sigma}{\sigma}\right)(c - h^{1-\alpha}(1-\tau_p\varphi)) \qquad (3.17)\\ \frac{\dot{h}}{h} &= c - h^{1-\alpha}(1-\varphi\tau_p) + h^{-\alpha}\varphi\tau_p(1-\eta). \qquad (3.18)\end{aligned}$$

[3] Note that only in the formulation of the transversality condition we use the $\star$ to denote optimal values.

Concerning a rest point of system (3.17) and (3.18) it should be noted that we only consider interior solutions. That means that we exclude the economically meaningless stationary point $c = h = 0$ such that we can consider our system in the rates of growth.[4] As to the existence and stability of a BGP we can state proposition 1.

Proposition 1 *If $1+\xi(1-\beta)(1-\sigma)/\sigma \geq 0$ there exists a unique BGP which is a saddle point.*

Proof: See appendix.

That proposition gives conditions for our model to be locally and globally determinate, i.e. there exists a unique value for $c(0)$ such that the economy converges to the BGP in the long run. A prerequisite for that outcome is $1+\xi(1-\beta)(1-\sigma)/\sigma \geq 0$. So, it can be stated that a unique BGP is the more likely the larger β and the smaller ξ for a given value of σ. From an economic point of view this means that an effective abatement technology, i.e. a large β, makes a unique BGP more likely. Further, a small ξ also favours that outcome. A small ξ implies that the effect of pollution on instantaneous utility is small. Thus, we can summarize that the smaller the negative external effect of production, either because abatement is very effective or because the household does not attach much value to a clean environment, the less likely is the emergence of global and local indeterminacy. So, it is the negative externality which gives rise to possible indeterminacy of equilibrium paths.

For given values of β and ξ, $1+\xi(1-\beta)(1-\sigma)/\sigma \geq 0$ always holds for $\sigma \leq 1$. As mentioned in the Introduction, the intertemporal elasticity of substitution, $1/\sigma$, often plays an important role as to the question of whether the model is global indeterminate. For our model we see from proposition 1 that $\sigma > 1$ is a necessary condition for multiple BGPs to be feasible. In other papers a small value for σ, i.e. a high intertemporal elasticity of substitution, is a necessary condition for multiple BGPs (see again for example Benhabib and Perli, 1994). The different outcome in our model compared to other contributions in the economics literature is due to the fact that in our model utility does not only depend on consumption but also on effective pollution which is a by-product of aggregate production.

Therefore, the outcome stated in proposition 1 makes sense from an economic point of view: Global indeterminacy means that the economy may either converge to the BGP with the high balanced growth rate or to the BGP with the low balanced growth rate in the long run. So it may either choose a path with a higher initial consumption level (but lower initial investment) or a path with a lower level of initial consumption (but higher initial investment). In the latter case, the household must be willing to forgo current consumption and shift it into the future. If production and, thus, consumption do not have negative effects in form of pollution then the household will do that only if he has a high intertemporal elasticity of substitution of consumption. However, if production and,

[4]Note also that h is raised to a negative power in (3.17).

thus, consumption do have negative repercussions because they lead to a rise in effective pollution (if $\beta < 1$) then the household is willing to forgo current consumption even with a low intertemporal elasticity of substitution because renouncing to consumption also has a positive effect since effective pollution is then lower, too, which raises current utility. It should be noted that our result is in line with the outcome in Koskela et al. (2000) who find that indeterminacy and cycles occur for relatively small values of the intertemporal elasticity substitution of consumption. But is must be recalled that their model is quite different from ours because it considers a renewable resource and no externalities and is formulated in discrete time.

Next, we consider the case $1 + \xi(1-\beta)(1-\sigma)/\sigma < 0$. Proposition 2 gives the dynamics in this case.

Proposition 2 *If* $1 + \xi(1-\beta)(1-\sigma)/\sigma < 0$, $\rho > -h_{min}^{-\alpha}\varphi\tau_p(1-\eta)\,(\sigma + \xi(1-\beta)(1-\sigma))/(1-\alpha)$ *is sufficient and necessary for the existence of two BGPs. The BGP yielding the lower growth rate is saddle point stable and the BGP giving the higher growth rate is asymptotically stable.*

Proof: See appendix.

This theorem states that two BGPs can be observed in our model depending on the parameter values and that one equilibrium is locally determinate while the other is indeterminate. One condition for this to hold is $1+\xi(1-\beta)(1-\sigma)/\sigma < 0$, meaning that σ must be larger 1. $\sigma > 1$ implies that $\xi(1-\beta) > 1$ must hold so that the inequality $1 + \xi(1-\beta)(1-\sigma)/\sigma < 0$ can be fulfilled. From an economic point of view, this means that pollution has a strong effect on utility, ξ is large, and abatement is not very effective, β is small.

It should be noted that the condition $\alpha \leq 0.5$ states that the capital share in the economy is less than 50 percent, which seems to hold true for real world economies.

4. Conclusion

In this paper we have analyzed the dynamics of an endogenous growth model with productive public spending and pollution. As to the financing of public spending we assumed that the tax on pollution is used for both abatement and for public investment, with the latter leading to sustained per capita growth.

We could demonstrated, without resorting to numerical examples, that the parameters determining the negative effect of pollution on utility is crucial as to the question of whether indeterminate equilibrium paths may exist as well as the intertemporal elasticity of substitution. In particular, it turned out that it is a small intertemporal elasticity of substitution consumption which is necessary for indeterminate equilibrium paths.

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Appendix

Proof of proposition 1: To prove proposition 1 we first calculate c^{∞} on a BGP which is obtained from $\dot{h}/h = 0$ as

$$c^{\infty} = h^{1-\alpha}(1 - \varphi\tau_p) - h^{-\alpha}\varphi\tau_p(1 - \eta).$$

Inserting c^{∞} in $\dot{c}/c$ gives

$$f(\cdot) \equiv \dot{c}/c = -\rho/\sigma + (1 - \varphi\tau_p)\alpha h^{1-\alpha}/\sigma - h^{-\alpha}\varphi\tau_p(1 - \eta)(1 + \xi(1 - \beta)(1 - \sigma)/\sigma).$$

For $(1 + \xi(1 - \beta)(1 - \sigma)/\sigma) \geq 0$ we have

$$\lim_{h \to 0} f(\cdot) = -\infty \ \text{ and } \ \lim_{h \to \infty} f(\cdot) = \infty \ \text{ and}$$

$$\partial f(\cdot)/\partial h = (1 - \varphi\tau_p)(1 - \alpha)\alpha h^{-\alpha}/\sigma + \alpha h^{-\alpha-1}\varphi\tau_p(1 - \eta)(1 + \xi(1 - \beta)(1 - \sigma)/\sigma) > 0.$$

This shows that there exists a finite $h^{\infty} > 0$ such that $f(\cdot) = 0$ holds and, thus, a unique BGP.

Saddle point stability is shown as follows. Denoting with J the Jacobian of $\dot{c}$ and $\dot{h}$ evaluated at the rest point we first note that $\det J < 0$ is a necessary and sufficient condition for saddle point stability, i.e. for one negative and one positive eigenvalue. The Jacobian in our model can be written as

$$J = \begin{bmatrix} c^{\infty}(1 + \alpha\xi(1 - \beta)(1 - \sigma)/\sigma) & c^{\infty}\phi \\ h^{\infty} & h^{\infty}\upsilon \end{bmatrix},$$

with ϕ given by $\phi = (1 - \varphi\tau_p)(1 - \alpha)(h^{\infty})^{-\alpha}(-1 + \alpha/\sigma) - (\xi(1 - \beta)(1 - \sigma)/\sigma)(1 - \alpha)\alpha(h^{\infty})^{-\alpha-1}[(h^{\infty})(1 - \varphi\tau_p) - \varphi\tau_p(1 - \eta)]$ and $\upsilon = -\alpha(h^{\infty})^{-\alpha-1}\varphi\tau_p(1 - \eta) - (1 -$

$\alpha)(h^\infty)^{-\alpha}(1-\varphi\tau_p)$. c^∞ and h^∞ denote the values of c and h on the BGP. The determinant can be calculated as $\det J = c^\infty h^\infty(-\alpha(1-\alpha)(1-\varphi\tau_p)(h^\infty)^{-\alpha}/\sigma) - \alpha(h^\infty)^{-\alpha-1}\varphi\tau_p(1-\eta)(1+\xi(1-\beta)(1-\sigma)/\sigma)) < 0$, for $(1+\xi(1-\beta)(1-\sigma)/\sigma) \geq 0$.

Proof of proposition 2: To prove proposition 2 we recall from the proof of proposition 1 that a h^∞ such that $f(\cdot) \equiv \dot{c}/c = 0$ holds gives a BGP.

For $(1+\xi(1-\beta)(1-\sigma)/\sigma) < 0$ we have

$$\lim_{h\to 0} f(\cdot) = \infty \text{ and } \lim_{h\to\infty} f(\cdot) = \infty \text{ and}$$

$$\partial f(\cdot)/\partial h >=< 0 \Leftrightarrow h >=< h_{min} \text{ and } \lim_{h\to 0} \partial f(\cdot)/\partial h = -\infty,\ \lim_{h\to\infty} \partial f(\cdot)/\partial h = 0,$$

with $h_{min} = (-1)\varphi\tau_p(1-\eta)(\sigma+\xi(1-\beta)(1-\sigma))(1-\alpha)^{-1}(1-\varphi\tau_p)^{-1}$.

This implies that $f(h,\cdot)$ is strictly monotonic decreasing for $h < h_{min}$, reaches a minimum for $h = h_{min}$ and is strictly monotonic increasing for $h > h_{min}$. This implies that there exist two BGPs (two points of intersection with the horizontal axis) if $f(h,\cdot)$ crosses the horizontal axis. This is guaranteed if $f(h_{min},\cdot) < 0$ holds. Inserting h_{min} in $f(\cdot)$ gives

$$f(h_{min},\cdot) = -\rho/\sigma - h_{min}^{-\alpha}\varphi\tau_p(1-\eta)\,(1+\xi(1-\beta)(1-\sigma)/\sigma)/(1-\alpha).$$

A necessary and sufficient condition for $f(h_{min},\cdot) < 0$ is $\rho > -h_{min}^{-\alpha}\varphi\tau_p(1-\eta)\,(\sigma+\xi(1-\beta)(1-\sigma))/(1-\alpha)$.

To analyze stability we note that the determinant of the Jacobian can be written as $\det J = -c^\infty h^\infty \partial f(\cdot)/\partial h$. This shows that the first intersection point of $f(\cdot)$ with the horizontal axis (smaller h^∞ and, thus, larger balanced growth rate, see (3.15)) cannot be a saddle point since $\partial f(\cdot)/\partial h < 0$ holds at this point. This point is asymptotically stable (only negative eigenvalues or eigenvalues with negative real parts) if the trace is negative, i.e. if $trJ < 0$ holds. The trace of the Jacobian can be calculated as $trJ = -h^\infty\varphi\tau_p + c^\infty\alpha(1+\xi(1-\beta)(1-\sigma)/\sigma)$ which is negative for $(1+\xi(1-\beta)(1-\sigma)/\sigma) < 0$.

The second intersection point of $f(\cdot)$ with the horizontal axis (lower h^∞ and, thus, higher balanced growth rate, see (3.15)) is a saddle point since $\partial f(\cdot)/\partial h > 0$ holds at this point implying $\det J < 0$.