

# Dynamic Consumption and Portfolio Decisions with Time Varying Asset Returns \*

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[preliminary version]

## Abstract

Recently research in financial economics has studied consumption and portfolio decisions, where investment opportunities change over time. This type of work originates in Merton (1971, 1990) who has used the Bellman equation to solve the consumption as well as asset allocation decisions for one state and two choice variables. Campbell and Viceira (1999, 2002) study consumption and portfolio decisions in various models with time varying expected returns by assuming that new investment opportunities are not only arising from changing interest rates, but also from time varying risk premia. They have approximated such a dynamic decision model under the assumption that the consumption-wealth ratio should not vary too much. In this paper, we study dynamic consumption and portfolio decisions by using dynamic programming which allows to compute, with sufficient accuracy, the decision variables and the consumption-wealth ratio at any point of the state space. The dynamic decision problem is first analytically and numerically solved for a simple model with constant returns. Then we solve a model with dynamic consumption and portfolio decisions when time varying returns are calibrated from the low frequency components of US time series financial data. The implications of the change of investor's risk aversion, the returns and the time horizon are explored. Finally, we solve a stochastic version of the model with mean-reverting returns.

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# 1 Introduction

In recent times seminal work has been undertaken to model dynamic consumption and portfolio decisions. Merton (1971, 1973) has provided a general intertemporal framework for studying the decision problem of a long term investor who not only has to decide about consumption and investment but also of how to allocate funds to assets such of equity, bonds and cash. It is now increasingly recognized that the static mean-variance framework of Markowitz needs to be replaced by a dynamic framework that takes into account new investment opportunities, different risk aversion of investors as well as their different time horizon.

There has been much effort that attempts to show that under certain restrictive conditions the dynamic decision problem is the same as for the static decision problem<sup>1</sup>, yet now it is well recognized that the more general dynamic framework starting with Merton (1971, 1973) is preferable. For the intertemporal model developed by Merton, however, one has difficulties to obtain closed form solutions. One thus needs numerical solution techniques to solve for the optimal consumption path and the asset allocation problem.

Campbell and Viceira (1999, 2002) use the assumption of log-normality of consumption and asset prices with the presumption that the optimal consumption-wealth ratio can be sufficiently approximated. Using log-linear expansion of the consumption-wealth ratio around the mean they can show the link between the myopic static decision problem and the dynamic decision problem, see Campbell and Viceira (2002, chs. 3 – 5).

They solve a simple model with time varying bond returns but with a constant equity premium.<sup>2</sup> They empirically calibrate the asset allocation problem for the US postwar period 1952.1 – 1999.4. Their exercise on the variation of the investor's risk aversion then shows that investors would increase their holdings of bonds and risk free asset (3 month bonds) as the risk aversion rises. Thus, long term investors who are risk averse tend to hold more bonds than equity.

Campbell and Viceira use a VAR approach<sup>3</sup> which allows for a time varying equity premium whereby the expected equity premium is driven by a forecasting variable such as the dividend-price ratio. They take up a proposition by Siegel (1994) that suggests that investing in equity with time varying risk premium, that, for example, follows a mean reversion process, is less risky than investing in bonds. Thus, the share of equity holding would be higher than for bonds in the long run. Siegel (1994:30) states:

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<sup>1</sup>See Samuelson (1963, 1969), Merton (1969), Fama (1970)

<sup>2</sup>See Campbell and Viceira (2002, ch. 3).

<sup>3</sup>See Campbell and Viceira (1999, 2002, ch. 4).

”Although it might appear riskier to hold stocks than bonds, precisely the opposite is true: the safest long-term investment has clearly been stocks, not bonds.”

If there is a predictable structure in equity (and bond) returns, and thus there are time varying expected returns, then clearly the optimal decision, with respect to consumption and portfolio rules, need to respond to the time varying expected returns.<sup>4</sup>

Since we here work with a deterministic model we approximate the time varying expected asset returns by the low frequency component of the returns.<sup>5</sup> On the other hand, recent theoretical research on asset pricing using loss aversion theory can give a sufficient motivation for such an assumption on time varying expected asset returns following a low frequency movement.<sup>6</sup> A further discussion of those issues is undertaken in appendix 2.

Yet, given such a low frequency movements of the returns, a buy and hold strategy for portfolio decisions will not be sufficient. Dynamic consumption decisions as well as a frequent rebalancing of the portfolio is needed in order to capture low frequency changes in returns and to avoid wealth and welfare losses.

In general, models with time varying returns are difficult to solve analytically and linearization techniques, as solution methods, for example a log-linear expansion about the equilibrium consumption-wealth ratio, as undertaken by Campbell and Viceira (2001, chs. 2-4) may be too inaccurate. This is likely to be the case if the consumption-wealth ratio is too variable.<sup>7</sup>

We use here a dynamic programming algorithm with flexible grid size that operates globally and can solve for any point in the state space, simultaneously for both the consumption decision as well as for the portfolio weights. As our more accurate solution technique shows the consumption-wealth ratio

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<sup>4</sup>There is plenty on empirical evidence of time varying expected returns; for early work see Campbell and Shiller (1988) and Fama and French (1988); for recent surveys, see Brandt (1999), Campbell and Viceira (1999) and Cochrane (2006).

<sup>5</sup>We are thinking here less of a univariate forecasting method such as mean reversion estimates, to obtain time varying returns but rather a multiple factor model to predict returns to be represented by low frequency movements of returns. As Cochrane (2006) argues the multivariate methods seem to perform better.

<sup>6</sup>For details see Barberis et al. (2001) and Grüne and Semmler (2005).

<sup>7</sup>In order to obtain an approximate solution of the model Campbell and Viceira (2002:51) presume that the consumption-wealth ratio is ”not too” variable. This procedure seriously loses accuracy with a parameter of risk aversion,  $\gamma > 1$ . Moreover, Campbell and Viceira use a model with a constant interest rate. See also Campbell (1993), Campbell and Viceira (1999) and Campbell and Koo (1997). On the issue of accuracy of using first and second order approximations, see Grüne and Semmler (2007).

can greatly vary and our solution remains sufficiently correct.<sup>8</sup> As previous work, we stay here with a power utility function of the investor.<sup>9</sup>

The remainder of the paper is organized as follows. In section 2 we illustrate the dynamic decision problem of consumption and asset allocation for a model with one asset and a constant return. We employ dynamic programming to study the impact of the variation of risk aversion, asset returns and time horizon on the dynamic paths of consumption, wealth accumulation and welfare. In section 3 we use a two asset model with two time varying returns and employ stylized facts on low frequency movements in asset returns, in particular for equity and a risk free asset in order to explore the above mentioned three issues. We calibrate the model with respect to US financial time series data. As alternative to taking low frequency movements in returns we employ in section 4 a model with mean reversion in returns. Here then we have to do this, of course, in the context of a stochastic version of the model. In sections 3 and 4 we also use dynamic programming as solution technique which allows for a more accurate solution for any point in the state space. Section 5 concludes the paper.

In the appendices a sketch of the dynamic programming algorithm is given, the data sources are discussed and a stochastic version of a dynamic portfolio problem illustrated.

## 2 Dynamic Consumption Decision: One Asset

We here first illustrate the use of the HJB<sup>10</sup> equation for a dynamic consumption choice problem, formulated in continuous time. We introduce a model with one asset and a constant return. The objective of the investor will here be to maximize his or her welfare given by a power utility function over consumption. We want to obtain the consumption choice for any point in the state space.

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<sup>8</sup>In Grüne and Semmler (2007) the out of steady state dynamics of 2nd order approximations and dynamic programming are compared. There it is shown that for 2nd order approximations the decision variables are approximately correct only in the vicinity of the steady state whereas the approximation of the value function shows bigger errors in the vicinity of the steady state as well as further away from it. The errors from dynamic programming are much smaller for both the decision variables and the value function and do not depend on the distance to the steady state.

<sup>9</sup>For a model of asset pricing with loss aversion, see Grüne and Semmler (2007).

<sup>10</sup>See Bellman (1967).

## 2.1 The Model

Our model is<sup>11</sup> a continuous time version of a dynamic decision problem. We study a choice problem that contains only one asset which generates a constant risk-free return. It could be thought of as a bond with a risk-free constant return. There is no choice between assets to be made, but only a choice of the optimal consumption path. We presume preferences over consumption of power utility type

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \quad (1)$$

There is only one asset,  $W$ , with a risk-free constant return  $r$ .

We presume that the agent maximizes the intertemporal discounted utility

$$\max_C \int_0^\infty e^{-\delta t} U(C_t) dt. \quad (2)$$

The wealth dynamic is given by

$$\dot{W} = rW - C \quad (3)$$

Using a dynamic programming approach (DP) leads to the following formulation

$$J = \max_{C_t} \int_0^\infty e^{-\delta t} U(C_t) dt \quad (4)$$

s.t. equ. (3) and  $W(0) = W_0$ .

The problem is to find the path  $C_t, t \geq 0$ , such that the objective function (2) obtains its optimal value.  $J$  is called the optimal value function, given the initial condition  $W(0) = W_0$ .

The HJB-equation for the DP problem (4) is (see Kamien and Schwartz 1997: 260)

$$-J_t(t, W) = \max_C \{e^{-\delta t} U(C) + J_W(t, W)(rW - C)\} \quad (5)$$

The first order condition for (5) is

$$e^{-\delta t} U'(C) - J_W(t, W) = 0 \quad (6)$$

Then using (1) for  $U$ , we get

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<sup>11</sup>This example was worked out by C.Y. Hsiao. We want to thank her for her effort.

$$J_W(t, W) = e^{-\delta t} C^{-\gamma}$$

Thus,

$$C = (J_W e^{\delta t})^{-\frac{1}{\gamma}} \quad (7)$$

Replacing  $C$  in (5) we obtain

$$\begin{aligned} 0 &= +J_t + e^{-\delta t} \frac{1}{1-\gamma} (J_W e^{\delta t})^{-\frac{1}{\gamma}(1-\gamma)} + J_W(rW - (J_W e^{\delta t})^{-\frac{1}{\gamma}}) \\ &= J_t + J_W rW + \frac{1}{1-\gamma} e^{\delta t(1-\frac{1}{\gamma}+1)} J_W^{1-\frac{1}{\gamma}} - J_W^{1-\frac{1}{\gamma}} e^{-\frac{\delta t}{\gamma}} \\ &= J_t + J_W rW + \frac{\gamma}{1-\gamma} (e^{-\frac{\delta t}{\gamma}} J_W^{1-\frac{1}{\gamma}}) = 0 \end{aligned} \quad (8)$$

Our guess for the value function is

$$J(t, W) = R(t) e^{-\delta t} U(W) = R e^{-\delta t} \frac{W^{1-\gamma}}{1-\gamma}$$

Then we obtain

$$J_t = -\delta J \quad (9)$$

$$J_W = \frac{1-\gamma}{W} J \quad (10)$$

$$\begin{aligned} e^{-\frac{\delta t}{\gamma}} J_W^{\frac{\gamma-1}{\gamma}} &= e^{-\frac{\delta t}{\gamma}} \left( R e^{-\delta t} W^{-\gamma} \right)^{\frac{\gamma-1}{\gamma}} \\ &= R^{\frac{\gamma-1}{\gamma}} e^{-\delta t} W^{1-\gamma} = (1-\gamma) R^{-\frac{1}{\gamma}} J \end{aligned} \quad (11)$$

One can check whether the above solution is the solution of our DP problem, by inserting (9), (10), (12) in (8) we obtain

$$\begin{aligned} -\delta J + r(1-\gamma)J + \gamma R^{-\frac{1}{\gamma}} J &= \\ J(rR^{-\frac{1}{\gamma}} + r(1-\gamma) - \delta) &= 0. \end{aligned} \quad (12)$$

If  $R$  satisfies  $(\cdot) = 0$  in (12) we have

$$R = \frac{\delta}{\gamma} + \frac{r(\gamma-1)}{\gamma}.$$

We get the solution for our DP problem

$$J(t, W) = \left( \frac{\delta}{\gamma} + \frac{r(\gamma - 1)}{\gamma} \right)^{-\gamma} e^{-\delta t} \frac{W^{1-\gamma}}{1-\gamma}. \quad (13)$$

Using (7) to get the optimal control

$$C^* = (J_W e^{\delta t})^{-\frac{1}{\gamma}} = R^{-\frac{1}{\gamma}} W.$$

Thus,

$$\frac{C^*}{W} = \frac{\delta}{\gamma} + \frac{r(\gamma - 1)}{\gamma}. \quad (14)$$

Our results for the steady state show that first, for this example indeed it holds that the consumption-wealth ratio is constant, second, the ratio increases in  $\delta$  (less patience), third, the ratio decreases in  $r$  (return effect), and fourth, consumption propensity is affected by  $\gamma$  (higher risk aversion). Yet, we do not know how the variables behave out of the steady state.

## 2.2 Numerical Results

A dynamic programming method, as sketched in the appendix 2<sup>12</sup>, can be used to study the out of steady state behavior of the variables and to compute the value function, the path of the control variable,  $C$ , the latter in feedback form from the state variable,  $W$ , and the path of the consumption wealth ratio.

In the numerical study of our model we take an interval  $\Omega = [0, 1] \times [0, 2]$  with grid points along each dimension. Later for  $\gamma \geq 1$  we choose  $\Omega = [0.0001, 1] \times [0.1]$ . As step size  $h$  we take  $h = \frac{1}{12}$ . As parameters we choose the constant return  $r = 0.03$ , risk aversion,  $\gamma = 0.75$ , and discount rate  $\delta = 0.06$ . The control  $C$  is also scaled by wealth, so we have  $c = \frac{C}{W}$ , the control space is  $\tilde{U} = [0, 0.7]$ , with  $q = 4001$  grid points.

Table 2.1 summarizes the chosen parameters and the gridding strategy.

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<sup>12</sup>See also Grüne (1997) and Grüne and Semmler (2004a)

Parameters	grid
$\Omega$	$[0, 1] \times [0, 1]$
$(n[0], n[1])$	(100,100)
$h$	$\frac{1}{12}$
$[t, T]$	$[0, 180]$
$r$	3%
$\gamma$	0,75
$\delta$	0.06

Table 2.1: Parameter Values and Grids

Figure 2.1 shows the numerically computed value function, which is concave due to the chosen utility function. As the figures 2.2a,b show the optimal consumption is a linear function of wealth, and the optimal consumption-wealth ratio converges toward to constant, with  $\frac{C}{W} = 0.07$ . As observable in figure 2.2b, the latter ratio is first not a constant but moves toward a constant.

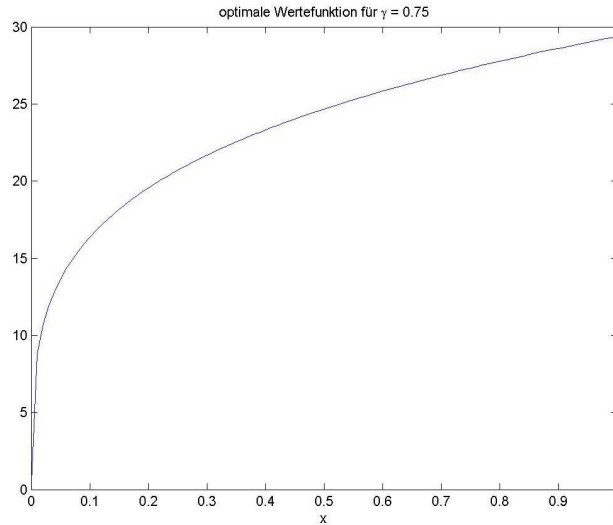


Figure 2.1: Value function in the interval  $\Omega = [0, 1]$



Figure 2.2a:

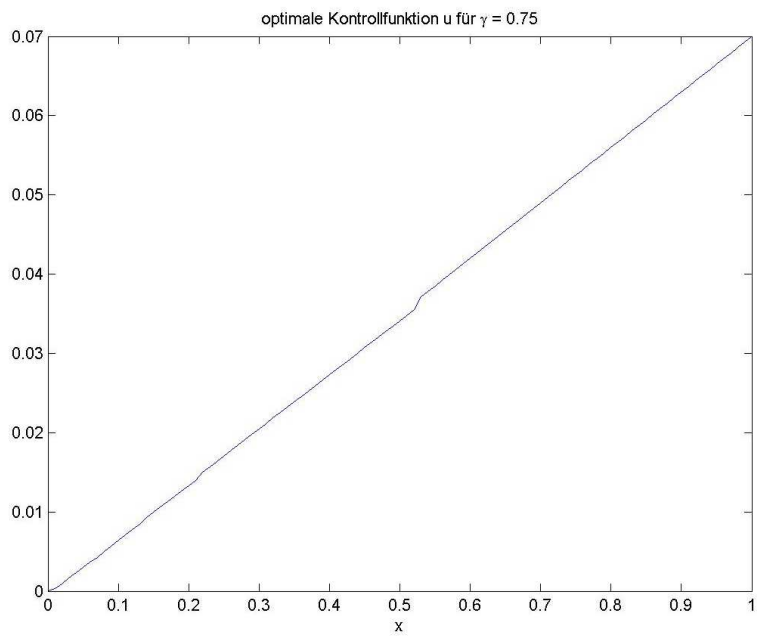


Figure 2.2b:

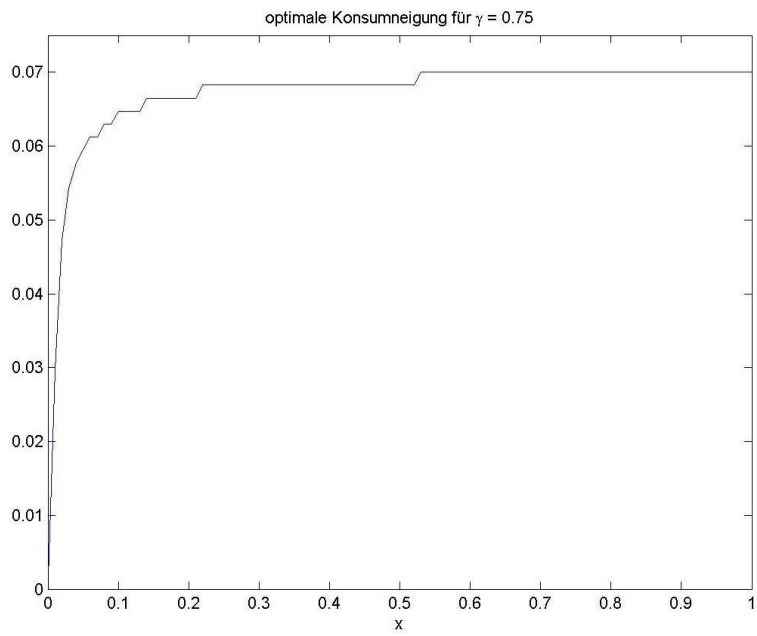


Figure 2.2a,b: Optimal consumption (upper figure), and consumption wealth ratio (lower figure), for the interval  $\Omega = [0, 1]$ .

Next, we explore the behavior of consumption and wealth for different initial level of wealth. With step-size  $h = \frac{1}{12}$ , we have assumed end time of  $T = 180$ . As figures 3.3 a,b show all trajectories, for different initial level of wealth, converge to  $W^* = 0$  in an asymptotically stable way. In addition, the smaller the initial wealth, the faster it is depleted.

Figure 2.3a:

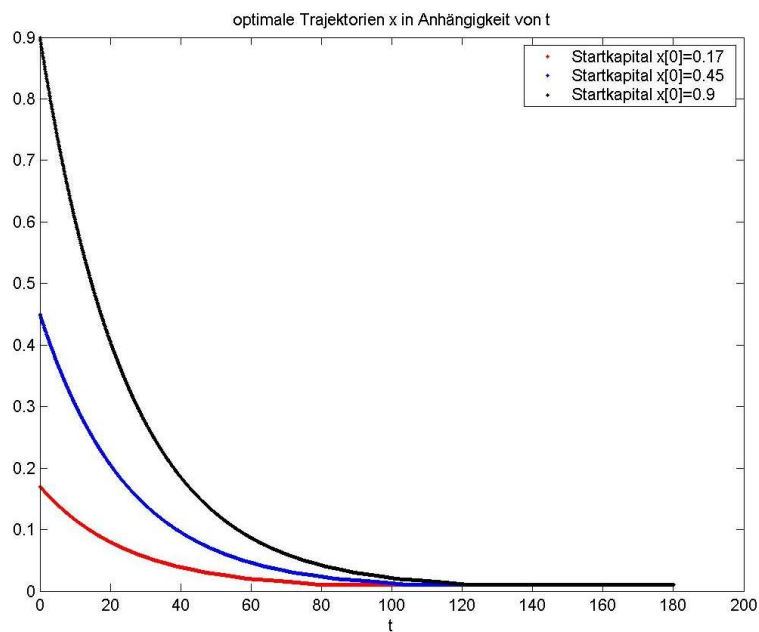


Figure 2.3b:

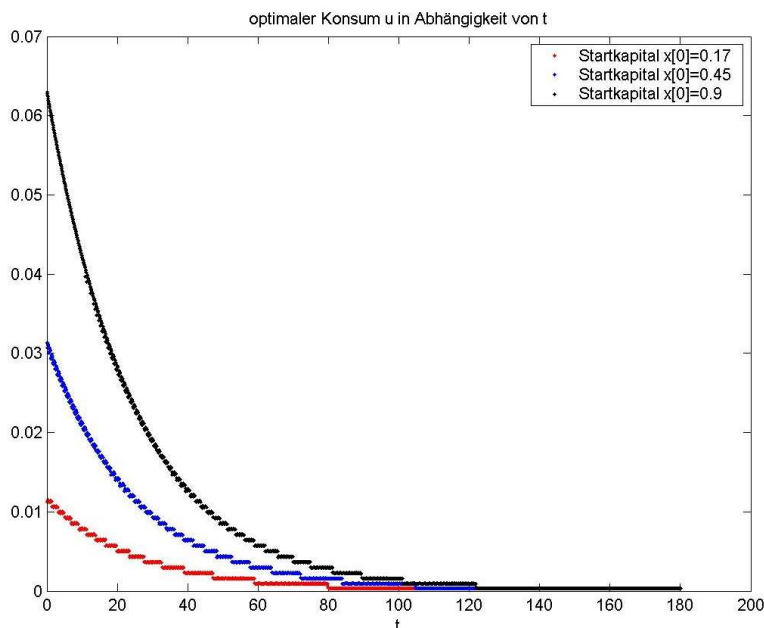


Figure 2.3a,b: Trajectories for wealth and consumption for different initial levels of wealth,  $[t, T] = [0, 180]$ .

A similar behavior is observable for the optimal consumption: it is lower, the lower the initial wealth. Moreover,  $C$  goes to zero at the same time as  $W$  goes to zero, depending however on initial wealth. Yet, note that in all of these exercises there are no multiple equilibria or bifurcations observable. Yet, overall in this simple example we already can observe that the out of steady level of the consumption-wealth ratio is not a constant.

## 2.3 Variation of Risk Aversion

As discussed in the introduction another important issue of portfolio theory is to explore of how optimal consumption and the path of wealth are affected by the attitude toward risk of the investors, in the power utility function represented by the parameter of risk aversion,  $\gamma$ . The more risk averse the investor is the more there is curvature in the utility function i.e. the higher the  $\gamma$ . Here we presume the same initial wealth for each investor. Table 2.2 captures the increasing risk aversion by the increasing  $\gamma$ .

Investors	Risk Aversion
A	$\gamma = 0.1$
B	$\gamma = 0.5$
C	$\gamma = 0.75$
D	$\gamma = 5$

Table 2.2: Investors' risk aversion

If we consider the value function, representing the welfare of the investor, we can observe in figures 2.4 a,b higher welfare for a higher risk aversion,  $\gamma$ .

Moreover, as figures 2.5a,b show consumption goes down with higher risk aversion and the consumption-wealth ratio also decreases with higher risk aversion parameter  $\gamma$ . The latter is also observable from equ. (14) for the equilibrium solution of the HJB equation.

Figure 2.4a:

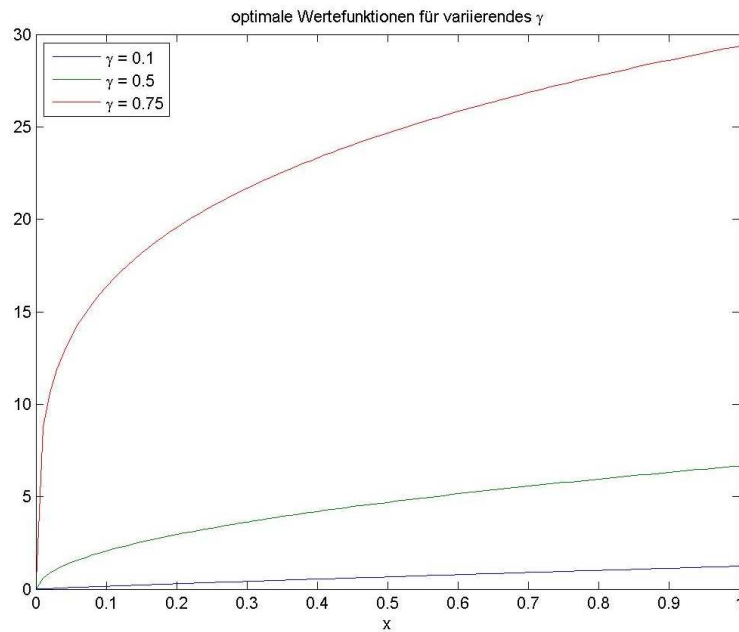


Figure 2.4b:

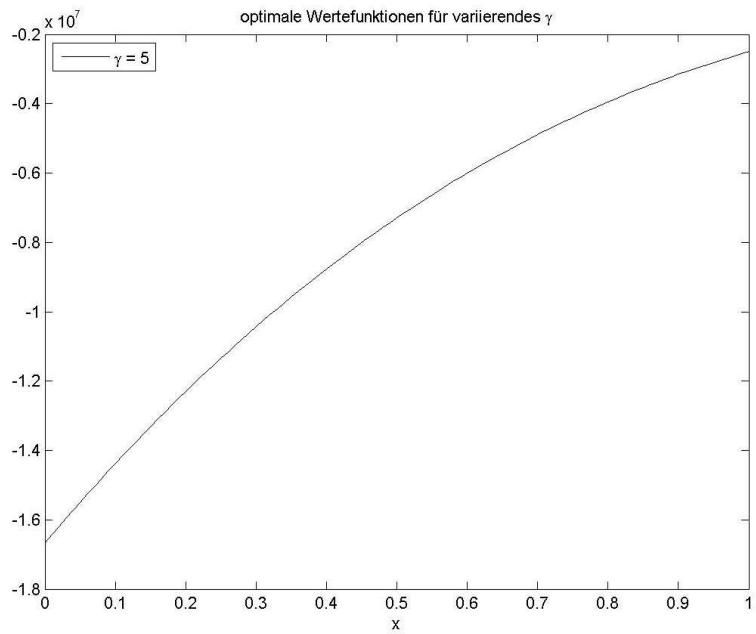


Figure 2.4a,b: Value function depending on risk aversion.

Figure 2.5a:

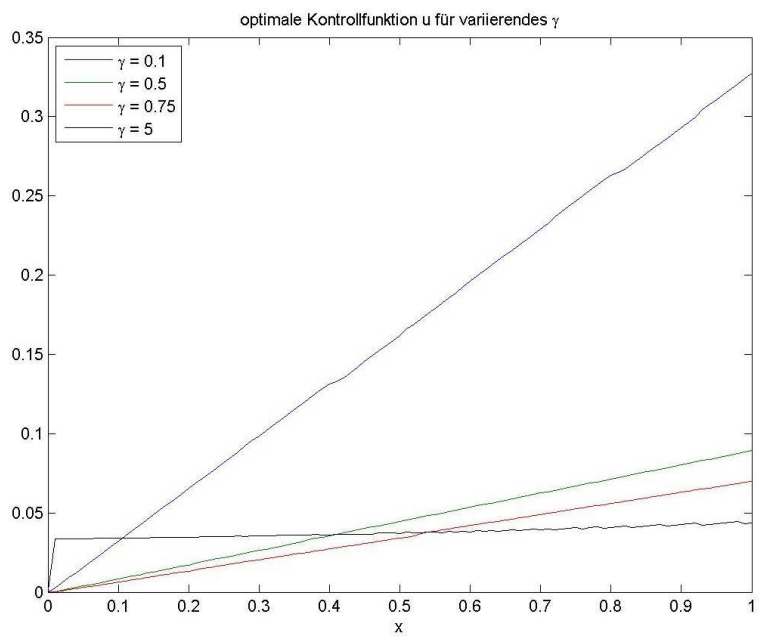


Figure 2.5b:

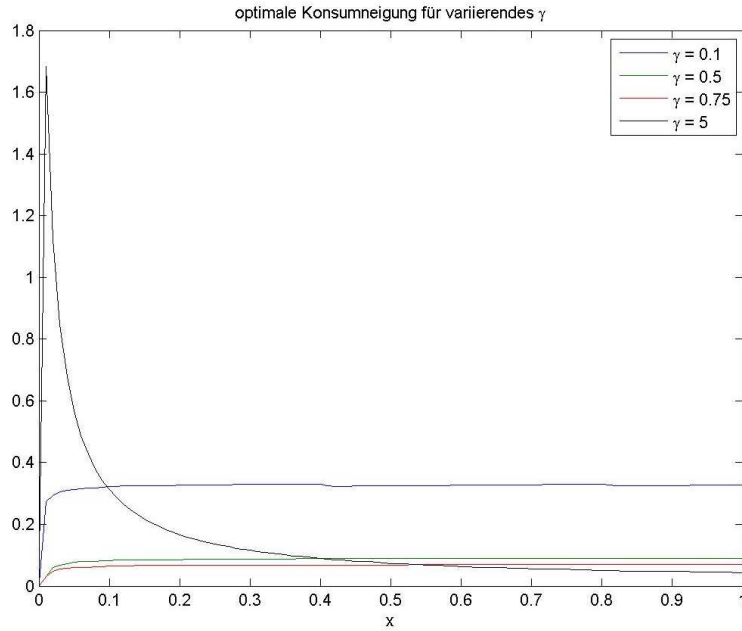


Figure 2.5a,b: Optimal consumption for  $\gamma= 0.1, 0.5, 0.75$  and consumption wealth ratio for discount rate  $\delta= 0.06$ .

Next, we present the vector field for the different risk aversion parameters,  $\gamma$ . The vector field analysis in figure 2.6 shows that the trajectories for all  $\gamma$ 's go to zero and thus wealth,  $W_t$ , goes to zero independently of the coefficient of relative risk aversion,  $\gamma$ .

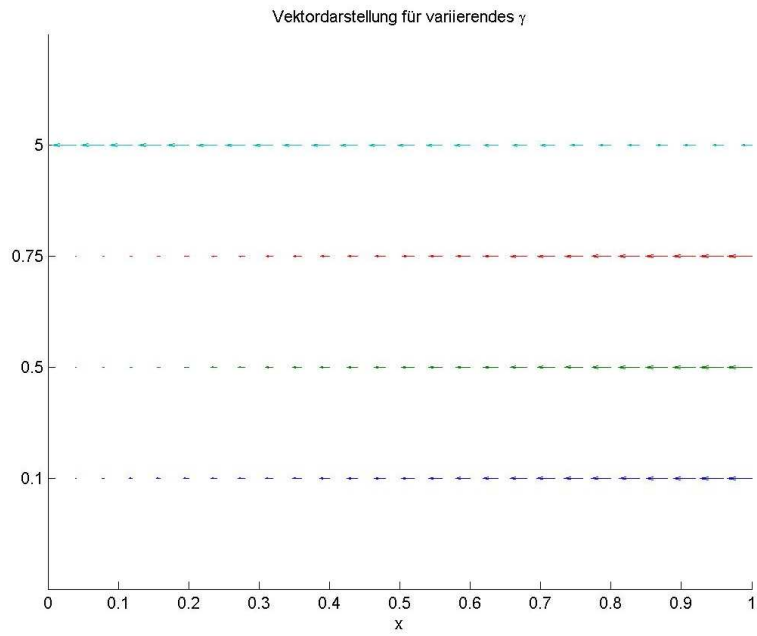


Figure 2.6: Vector fields for  $\gamma = 0.1, 0.5, 0.75$  and  $5$  and, discount rate  $\delta = 0.06$ .

## 2.4 Variation of Returns

Next we want to explore the effects of the variation of asset returns on the value function, consumption choice and the dynamic of wealth. The asset returns are chosen as described in table 2.2.

Asset	Asset Returns
A	$r = 3\%$
B	$r = 5\%$
C	$r = 6\%$
D	$r = 9\%$

Table 2.2: Asset Returns

As one can observe from figure 2.7, the value function increases in size with the size of the asset return.

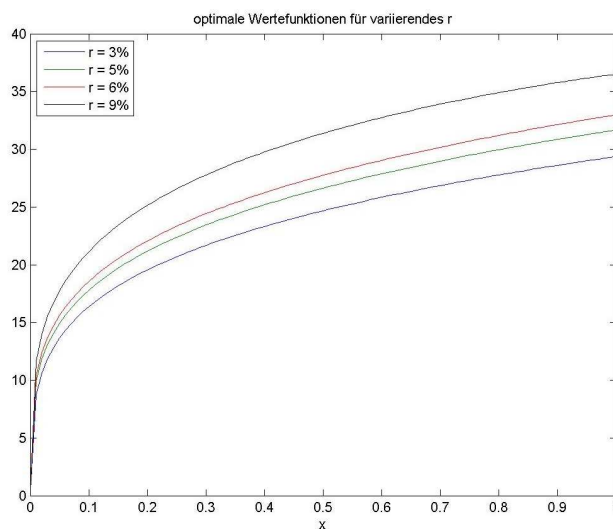


Figure 2.7: Value function for asset returns  $r = 3\%, 5\%, 6\%$  and  $9\%$  with  $\delta = 0.06, \gamma = 0.75$  and  $\delta = 0.05$ .

Further, figure 2.8 shows that consumption is proportional to wealth. Figure 2.8a,b demonstrates that the consumption-wealth ratio is first not a constant but converges toward constant, except for the highest return of



$r = 9\%$ . In fact, the consumption wealth ratio becomes non-stationary for  $r > 6\%$ .

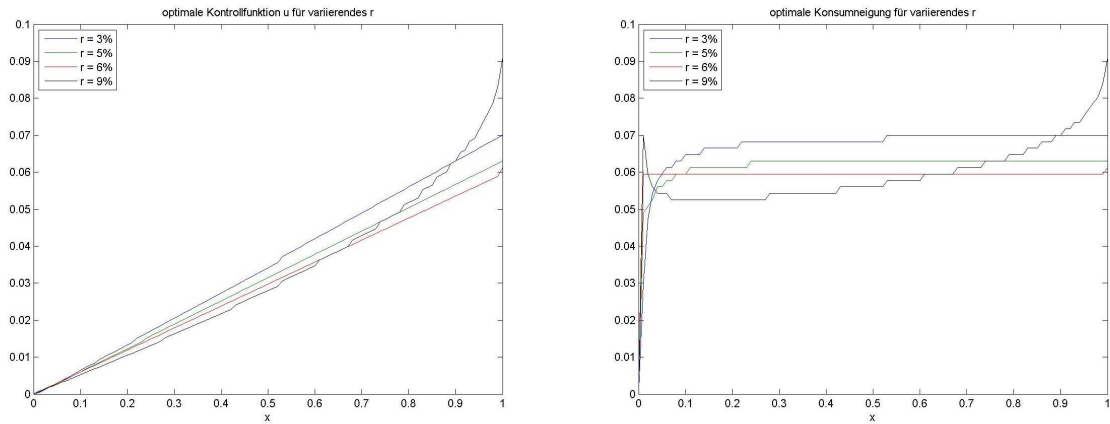


Figure 2.8a,b: Optimal consumption and consumption wealth ratio  
 $r = 3\%$ ,  $5\%$ ,  $6\%$  and  $9\%$  discount rate  $\delta = 0.06$ .

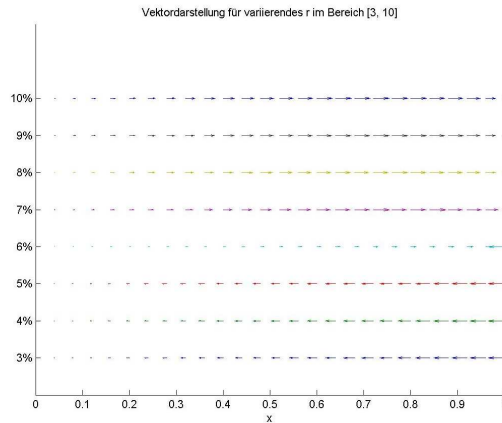


Figure 2.9: Vectorfield for  $r \in [3, 10]$

This result is also confirmed by the next figure. As the vector field of figure 2.9 for the asset returns  $r \in [3, 10]$  shows, for an  $r < 6\%$  the wealth shrinks whereas for an  $r \geq 6\%$  the wealth expands. Note that we have undertaken the exercise here for a constrained state space. What we wanted to explore here was the bifurcation of the asset dynamics occurring at  $r = 6\%$ .<sup>13</sup>

<sup>13</sup>We do not explore further whether there might be new positive stationary states  $W > 0$ .

## 2.5 Variation of Time Horizon

Next we explore the effect of the time horizon on the value function, consumption and dynamic of wealth.

	Discount Rates
A	$\delta = 0.01$
B	$\delta = 0.03$
C	$\delta = 0.06$
D	$\delta = 1$

Table 2.3: Used discount rates,  $\delta$ .

As one would expect from economic theory, the value function becomes less steep the higher the discount rate is, see figure 2.10. With large a discount rate the asset holder prefers consumption in the near future over consumption further away from the current time period. The agent has thus a preference to run down his or her wealth with a higher discount rate.

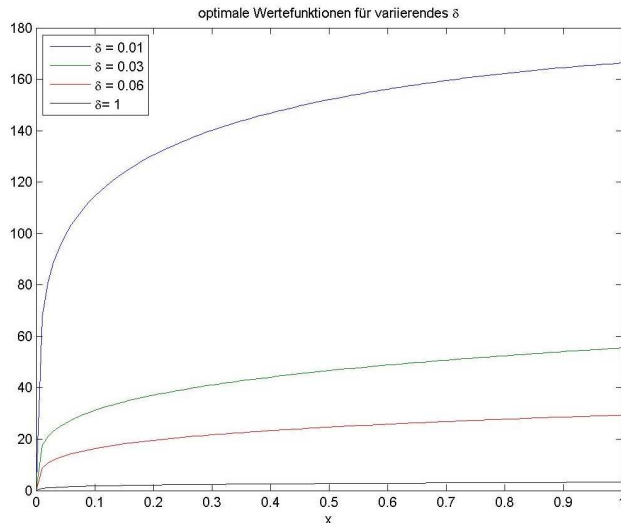


Figure 2.10: Discount Rates 0.01, 0.03, 0.06 and 1.

The fact that consumption another consumption wealth ratio is rising with the discount rate, is also demonstrated in figures 2.11 a,b,c,d.

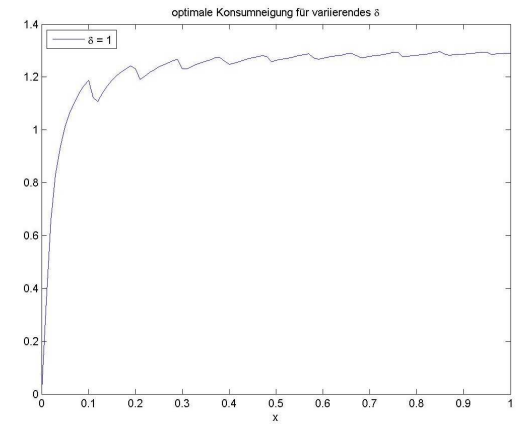
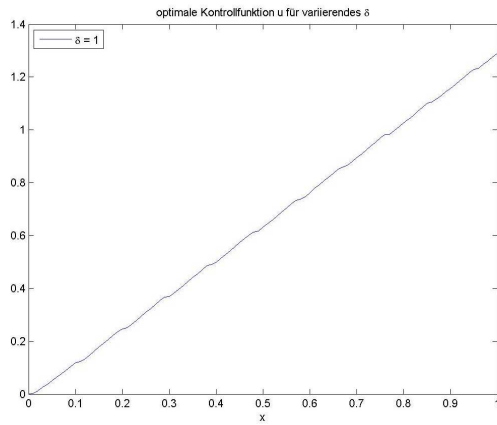
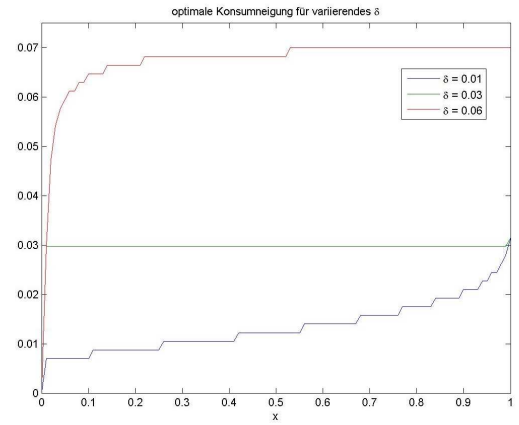
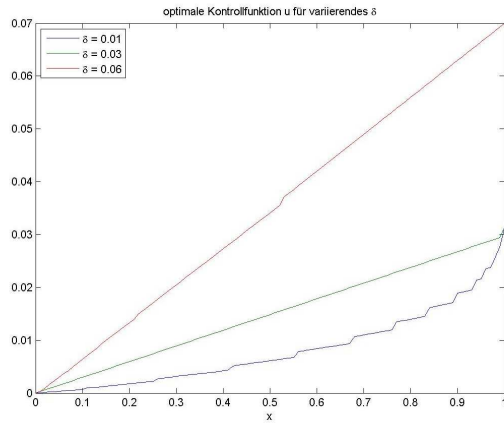


Figure 2.11a,b,c,d: Consumption (a,c) and consumption-wealth ratio (b,d) for discount rates 0.01, 0.03, 0.06 and 1.

The figure 2.12 shows that there is, as for the asset returns in section 2.4, also a bifurcation of the wealth dynamics, here for a discount rate of roughly  $\delta = 0.0035$ .

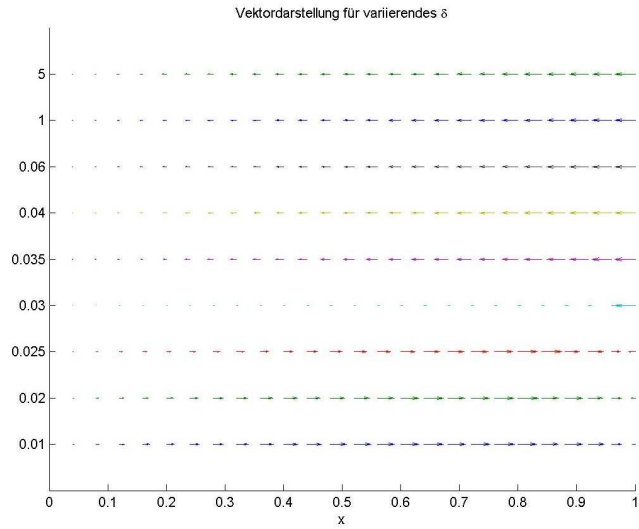


Figure 2.12: Bifurcation of wealth dynamics, region  $[0, 5]$ .

Overall, in our model with one asset and a constant return we can mostly observe results that we can expect from theory. Yet, for all parameter constellations the consumption-wealth ratio is not a constant outside the steady state. First, it is far away from its steady state ratio, but then converges toward some constant (except in some cases where we see, due to some bifurcation behavior, a divergent behavior).

### 3 Dynamic Consumption and Portfolio Decisions: Two Assets and Time Varying Returns

Next we are considering two assets. Both of them are characterized by predictable returns that can be represented by low frequency movements in their returns. Those asset returns may be viewed as time varying mean returns. Here, we concentrate on dynamic portfolio decisions for a deterministic version of the model. Therefore, our low frequency components take the place of time varying expected returns.<sup>14</sup> We want to undertake a study of dynamic consumption and portfolio decisions for the case when the returns follow such a low frequency movement. For our propose it is sufficient to presume that actual financial time series data can be decomposed into two different time scales: a low and high frequency movement. The appendix 2 of the paper presents empirical results of stock returns, interest rates and bond returns that are decomposed into a high and low frequency movements, using an appropriate filter. There, we also briefly discuss to what an extent low frequency components of asset returns can successfully be employed in forecasting returns.

By nature consumption and portfolio decisions are not decisions on high frequency data, but are rather based on low frequency movements of the data. The financial market practioners, for example, dynamically rebalance portfolios by looking at low frequency movements in the financial data. That is what we attempt to model in our next model variant, whereby we simply presume that the filtered process will generate a low frequency component of the returns. In the model below, we approximate this by an appropriate sine-wave function.

#### 3.1 The Model with Time Varying Returns

Let us introduce a model with two assets and two returns whereby the returns follow a deterministic low frequency movement. We presume here that thus can be stylized as a sine wave function depending on time. Empirical evidence that time series data on returns follow a sign wave function is provided in appendix 2. Moreover, we also assume that there might be a phase shift as concerning the two wave functions used here. We concentrate on two returns with low frequency movement, representing the equity return and another one the short-term interest rate. The empirical evidence, reported

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<sup>14</sup>The latter is what one would have in a stochastic version of the model.

in appendix 2, also seems to suggest such a behavior of those two returns.

Our dynamic portfolio decision problem can be stated as

$$\max_{\{c^*, \alpha\}} \int_0^\infty e^{-\delta t} U(C_t) dt \quad (26)$$

$$\text{s.t. } \dot{W}(t) = \alpha_t R_{e,t} W_t + (1 - \alpha_t) R_{f,t} W_t - C_t \quad (27)$$

$$\dot{x}(t) = 1. \quad (28)$$

Hereby we presume that the mean of the returns for the short term interest rate,  $R_{f,t}$  and the equity return,  $R_{e,t}$  are time dependent and can be formulated as

$$R_{f,t}(x_t) = \alpha_1 \sin(\alpha_3 x_t) \quad (29)$$

$$R_{e,t}(x_t) = \alpha_2 \sin(\alpha_4 x_t) + \alpha_5 \quad (30)$$

In the following  $\alpha_3$  is presumed to be a multiple of  $\alpha_4$ , or the reverse. For  $\alpha_4 = k \cdot \alpha_3$  and  $x_t = b_2$  then holds

$$\alpha_3 x_t = \alpha_3 b_2 = j_1 2\pi$$

$$\alpha_4 x_t = \alpha_4 b_2 = j_2 2\pi$$

and thereby either  $j_1 = 1, j_2 = k$  or  $j_2 = 1, j_1 = k$ . In case  $j_1 = 1$  holds, we have

$$b_2 = \frac{2\pi}{\alpha_3}$$

On the other hand, if  $j_2 = 1$  holds, it follows

$$b_2 = \frac{2\pi}{\alpha_4}$$

Parameter	Parameter Values
$\Omega$	$[0, 350] \times [0, \frac{2\pi}{\alpha_3}]$ and $[0, 350] \times [0, \frac{2\pi}{\alpha_4}]$
$(n[0], n[1])$	$(350, 30)$
$(W[0], W[1])$	$(W_1, x_2) \in \Omega$
$h$	$\frac{1}{12}$
$[t, T]$	$[0, 90]$
$N0$	8
$N1$	1
$\gamma$	0.,75
$\delta$	0.05
$\alpha_1$	0.1
$\alpha_2$	0.2
$\alpha_3$	0.2
$\alpha_4$	0.2
$\alpha_5$	0.005

Table 3.1. Parameter values for the model.

As in the previous section, it is worth exploring the role of the degree of risk aversion of the investor,  $\gamma$ , the variation of returns,  $R_{e,t}$  and  $R_{f,t}$ , and the time horizon an investor has when making on investment and consumption decisions. We will study the effects on the value function, the paths of consumption and the consumption-wealth ratio, the vector fields and the optimal trajectories. Before, however, those variants are explored we study a benchmark case.

### 3.2 Benchmark Case

Let us first study again the case where we keep the parameters fixed but explore the role of initial conditions. Note that in the following  $x_2 = t$  and  $x_1 = W$ . Thus we can observe a more wave like movement of consumption and the consumption-wealth ratio, the higher the initial wealth,  $W$ . In other words, the former two are depend on the size of the initial asset, see figures 3.1 and 3.2.

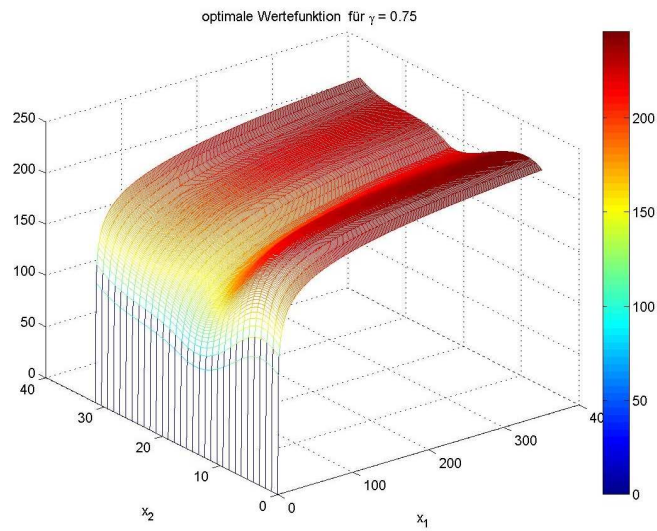


Figure 3.1: Value function for  $\gamma = 0.75$  und  $\delta = 0.05$ ,  $W$

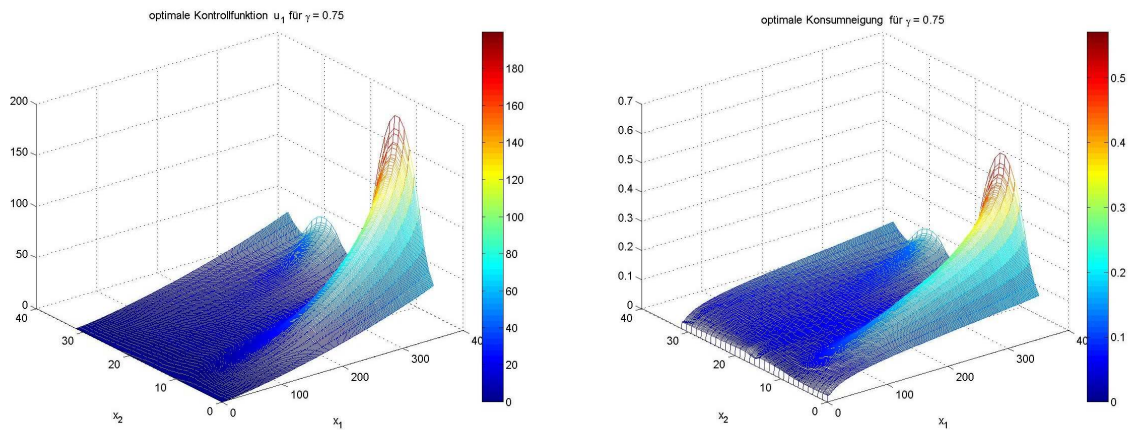


Figure 3.2: Optimal consumption (left) and consumption wealth ratio (right),  $\gamma = 0.75$  and  $\delta = 0.05$ .

In figure 3.3 the vector field shows the dynamics of the consumption portfolio choice: for low wealth and a small number of time steps, asset value rises, up to 340 and then falls.



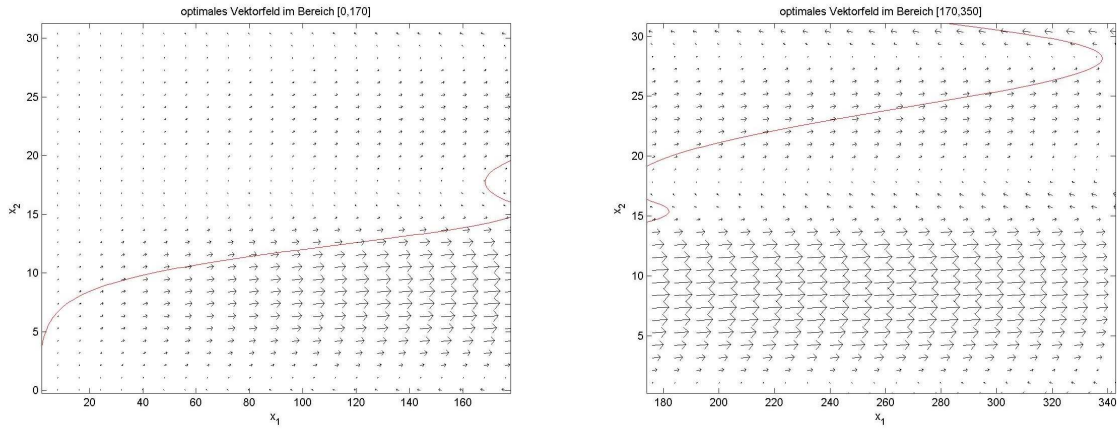


Figure 3.3: Vector field and trajectories for  $W_0 = (1, 0)$  for  $\gamma = 0.75$ .

Figure 3.4 shows the dependence of the movement of wealth, consumption and the portfolio weight,  $\alpha$ , on time. Hereby the upper line is wealth,  $W$ , then follows consumption, and the (jumping) straight line is  $\alpha$ . We can observe, if there is a positive equity premium,  $\alpha$  will be positive, otherwise  $\alpha$  is negative. Note that we have treated here the decision maker to be constrained. We presume  $-3 \leq \alpha \leq 4.5$ ,

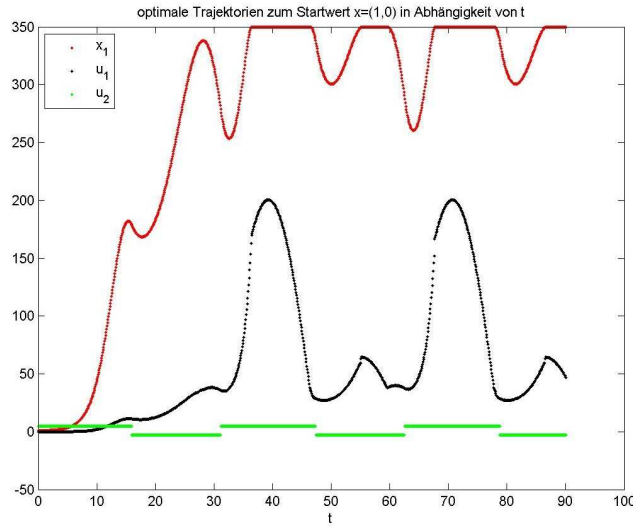


Figure 3.4: Optimal trajectory for  $W_0 = (1, 0)$  for  $\gamma = 0.75$ .

### 3.3 Variation of Risk Aversion

One of the important issues in dynamic portfolio decisions is how consumption decisions and portfolio weights are affected by the risk aversion of the investor. In our power utility function which, we have used here, investors with a lower (higher) risk aversion are characterized by a lower (higher)  $\gamma$ . Note here again  $x_2 = t$  and  $x_1 = W$ . In the figures 3.5 we can observe that the value function is smaller but also becomes flatter as risk aversion rises.

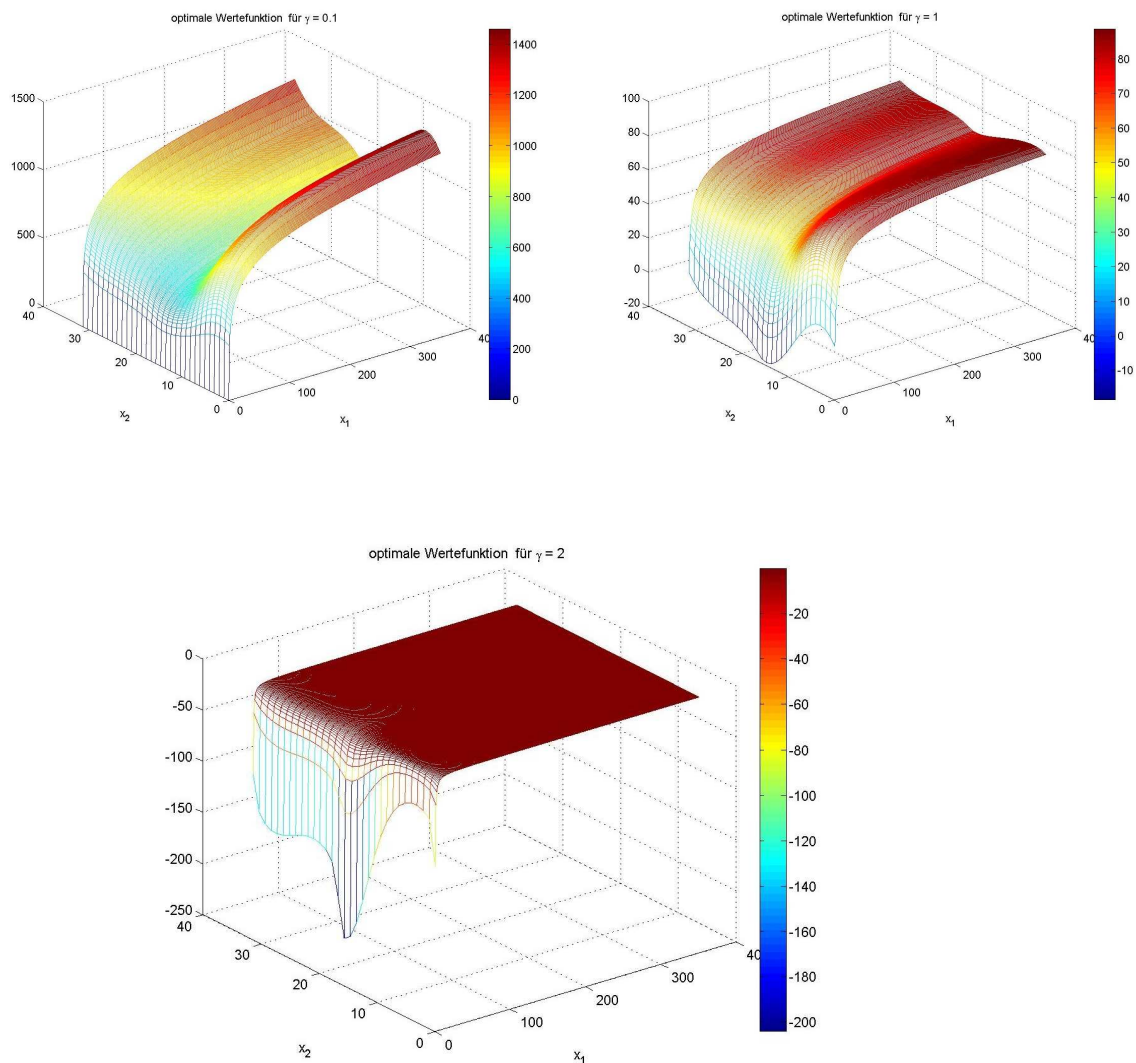


Figure 3.5: Value function for  $\gamma : 0.1$  (upper left),  $\gamma = 1$  (upper right)  $\gamma = 2$  (below)

The next figure, figure 3.6, shows that the larger the risk aversion, the higher is the optimal consumption, see lower panel of figure 3.6, with  $\gamma = 2$ . Yet, it also fluctuates more, see lower panel with  $\gamma = 2$ .

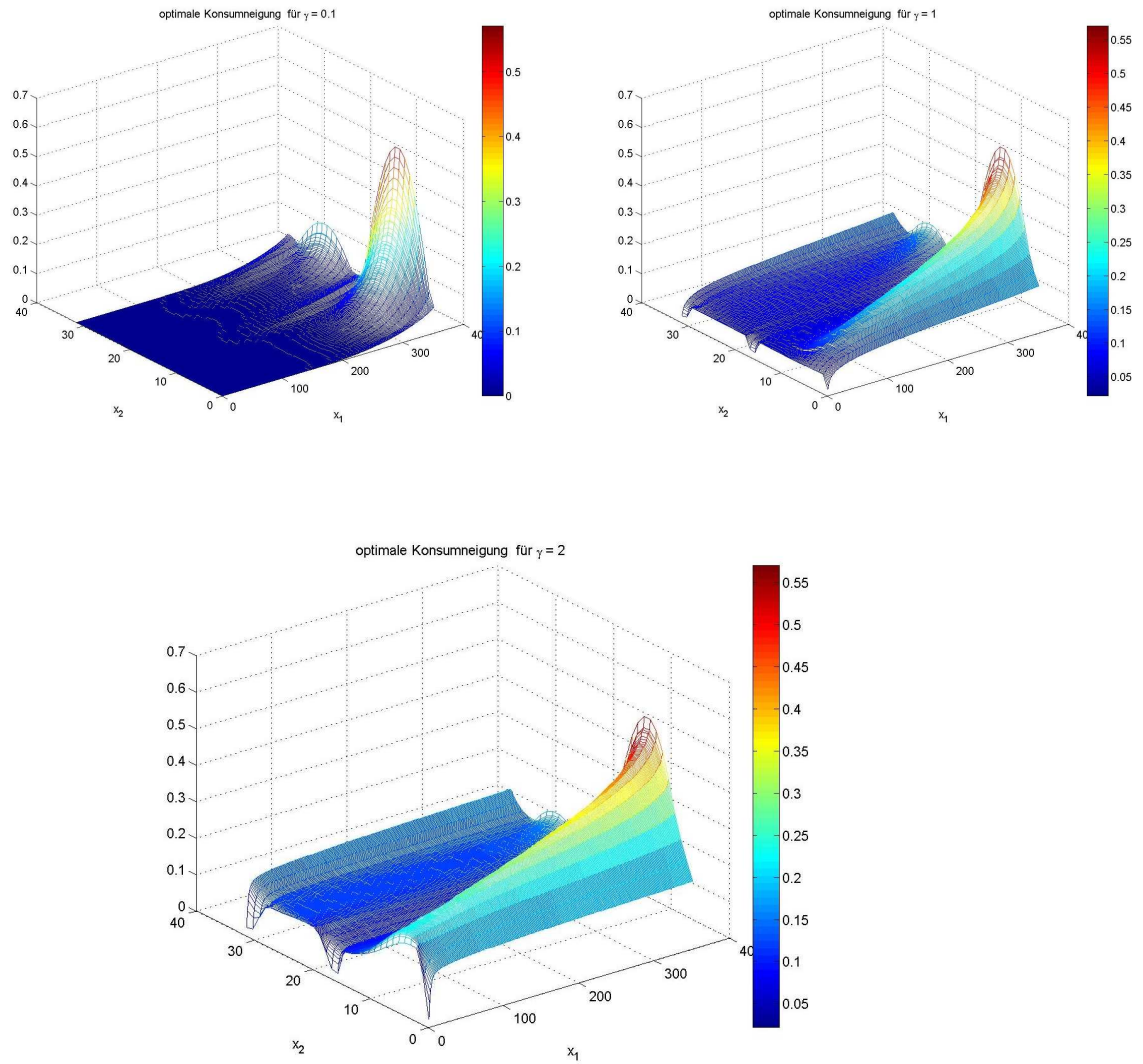
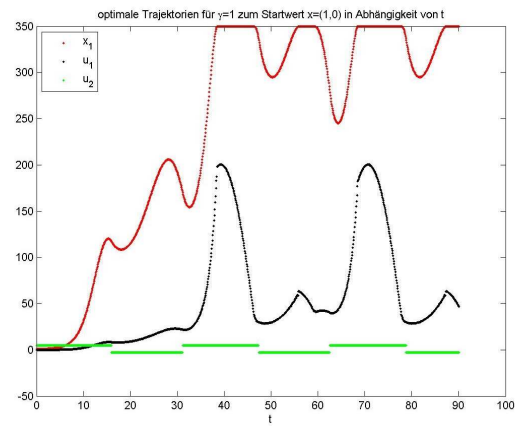
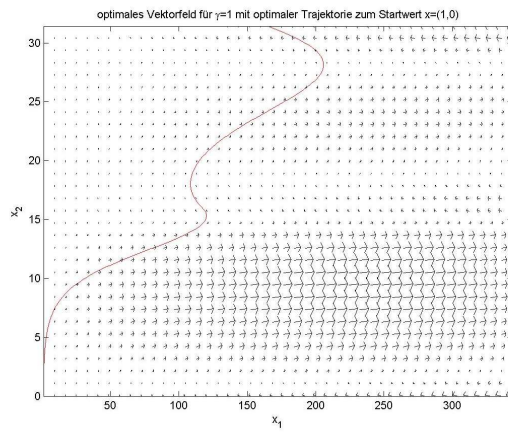
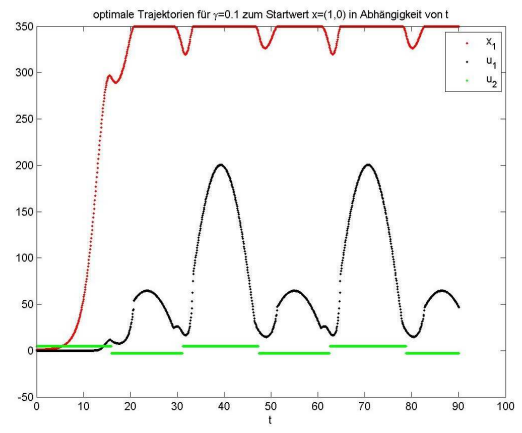
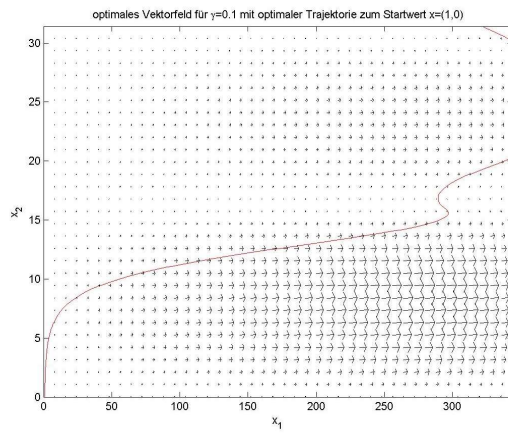


Figure 3.6: Consumption-wealth ratio for  $\gamma : 0.1$ , (upper left)  $\gamma = 1$  (upper right)  $\gamma = 2$  (below)

Moreover, in figure 3.7 we can observe, the lower the risk aversion, the more rapid the asset is built up, see right figures from above to below.



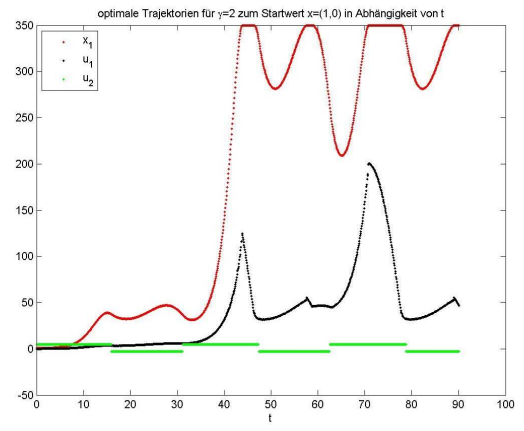
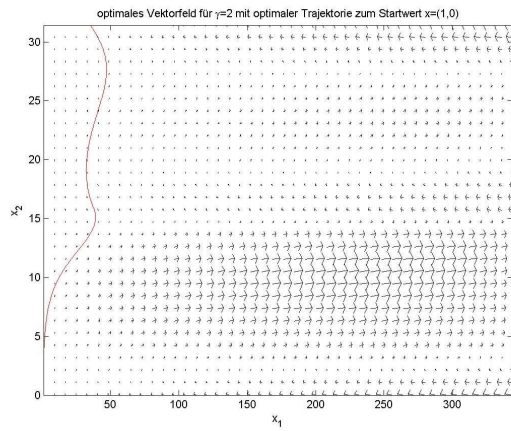


Figure 3.7: Vector fields (left) for  $\gamma : 0.1, 1$  and  $2$  (from above to below) and corresponding trajectories

### 3.4 Variation of Returns

Next, we explore the role of the fluctuation in the magnitude of the returns for dynamic asset allocation decisions. The movement of the asset returns,  $R_{e,t}$  and  $R_{f,t}$  are given by the parameters presented in table 3.2. There three variants are proposed.

**Table 3.2: Parameters for the returns,  $R_{e,t}, R_{f,t}$**

	<b>Benchmark</b>	<b>Variant 1</b>	<b>Variant 2</b>	<b>Variant 3</b>
$\alpha_1$	0.1	0.1	0.1	0.1
$\alpha_2$	0.2	0.2	0.2	0.2
$\alpha_3$	0.2	0.4	0.8	0.2
$\alpha_4$	0.2	0.2	0.2	0.4
$\alpha_5$	0.005	0.005	0.005	0.005

Figure 3.8 shows that the more there are waves in the returns, the more there are waves in the value function.

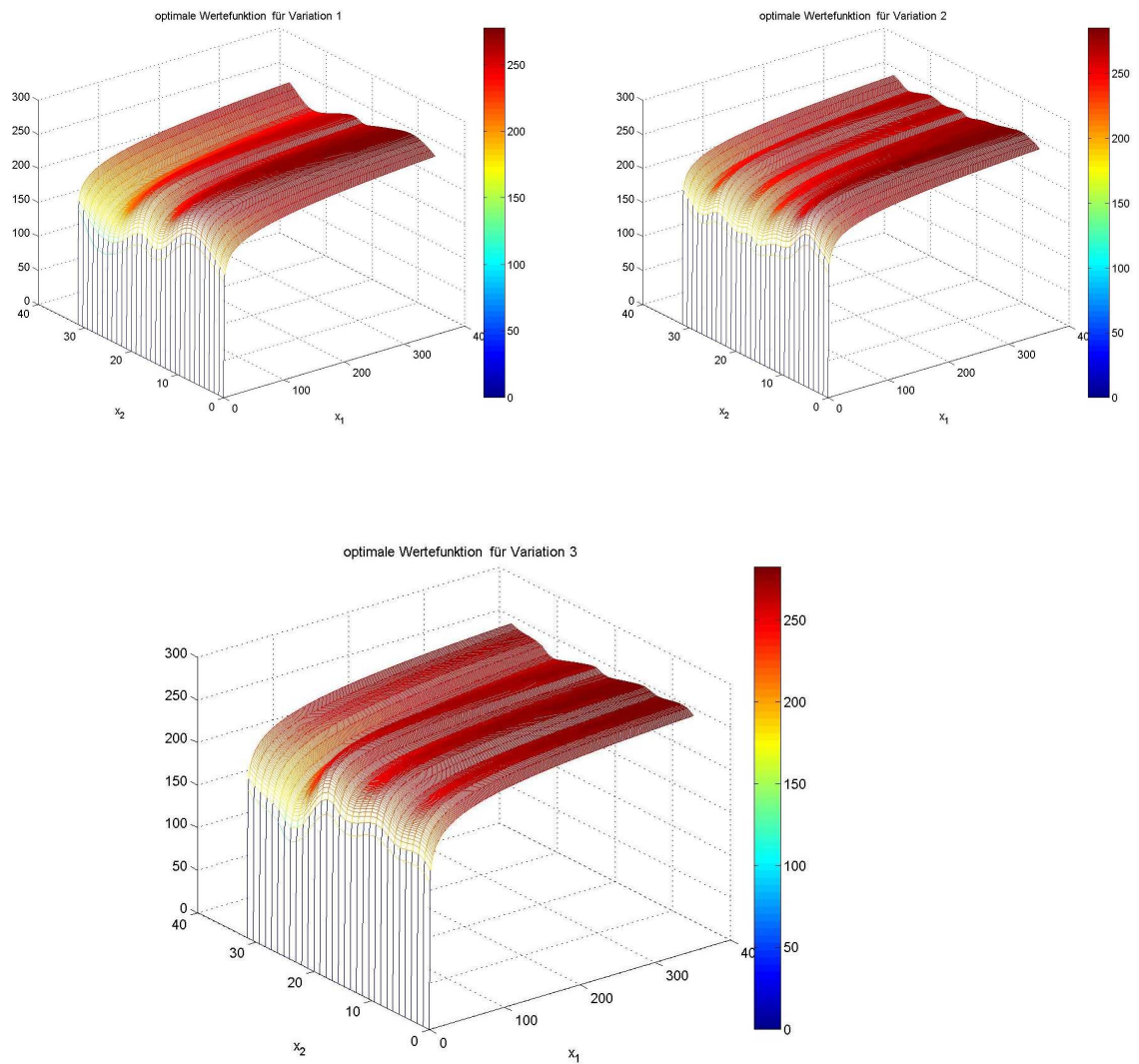
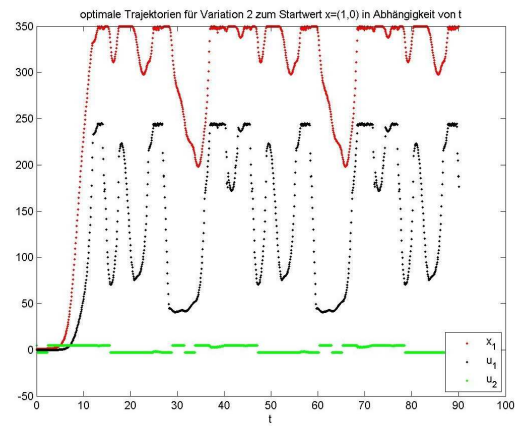
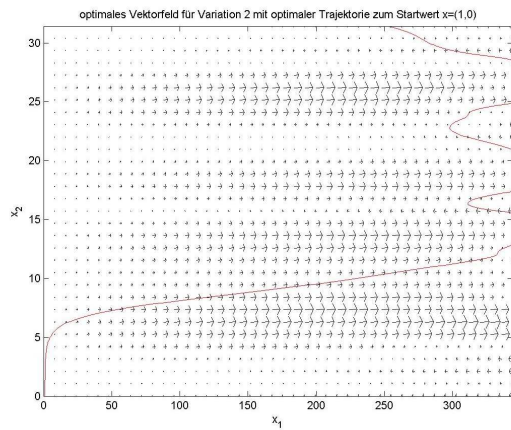
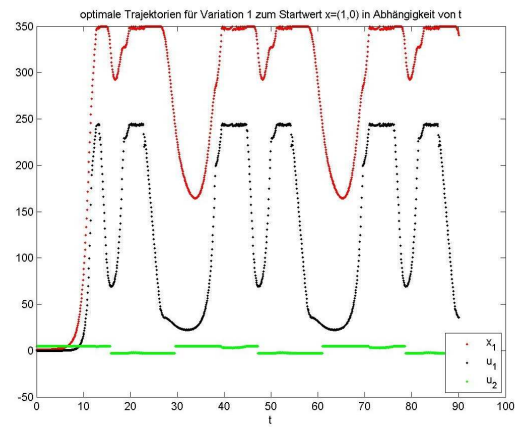
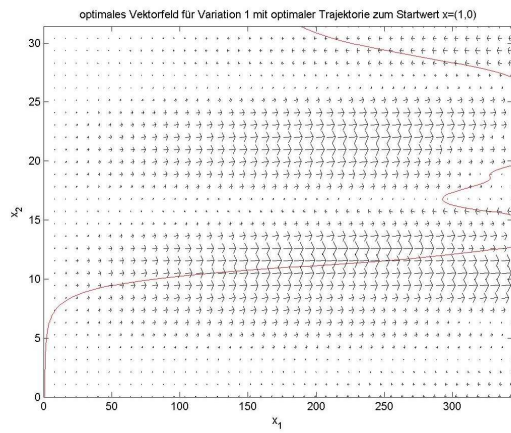


Figure 3.8: Value function for variation of returns,  $R_{e,t}$ ,  $R_{f,t}$  (see table 3.2). Variant 1 (upper left), variant 2 (upper right), variant 3 (below).

This is also shown in the figure 3.9 for the swings in wealth accumulation, consumption and the portfolio weight,  $\alpha$ .





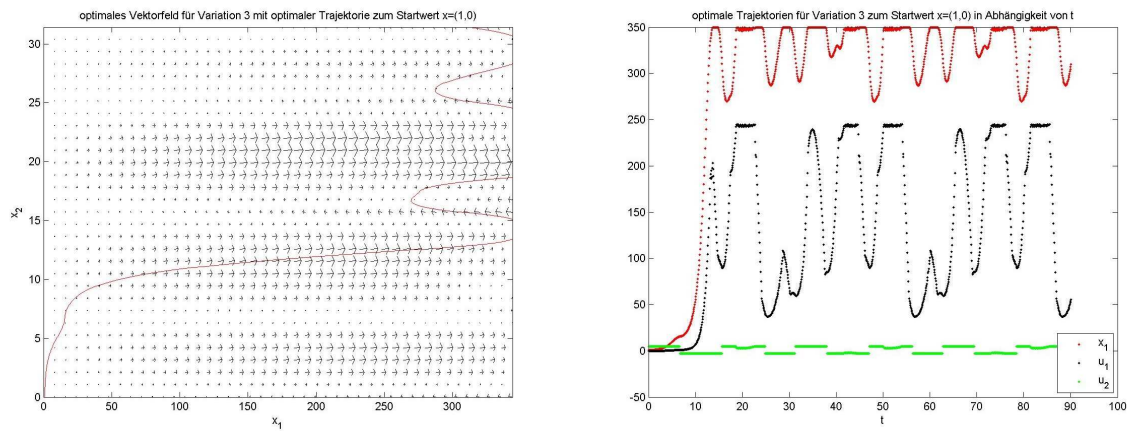


Figure 3.9: Vector fields and optimal trajectories for variation in returns  $R_{e,t}$  and  $R_{f,t}$ . Variant 1 (upper panel), variant 2 (middle panel), variant 3 (lower panel)

### 3.5 Variation of Time Horizon

Finally, we want to explore the role of the time horizon for dynamic asset allocation decisions. Here we vary the discount rate whereby a low (high) discount rate,  $\delta$ , represents a long (short) time horizon. We take,  $\delta = 0.1$ ,  $\delta = 1$  and  $\delta = 11$ .

As one can observe, the wave-like behavior of the value function is reduced, the shorter the time horizon is for the investor. Thus, dynamic optimization over a longer time horizon, will make the fluctuation of the value function stronger. Moreover, as one would expect from theory, the size of the welfare shrinks with a higher discount rate,  $\delta$ .

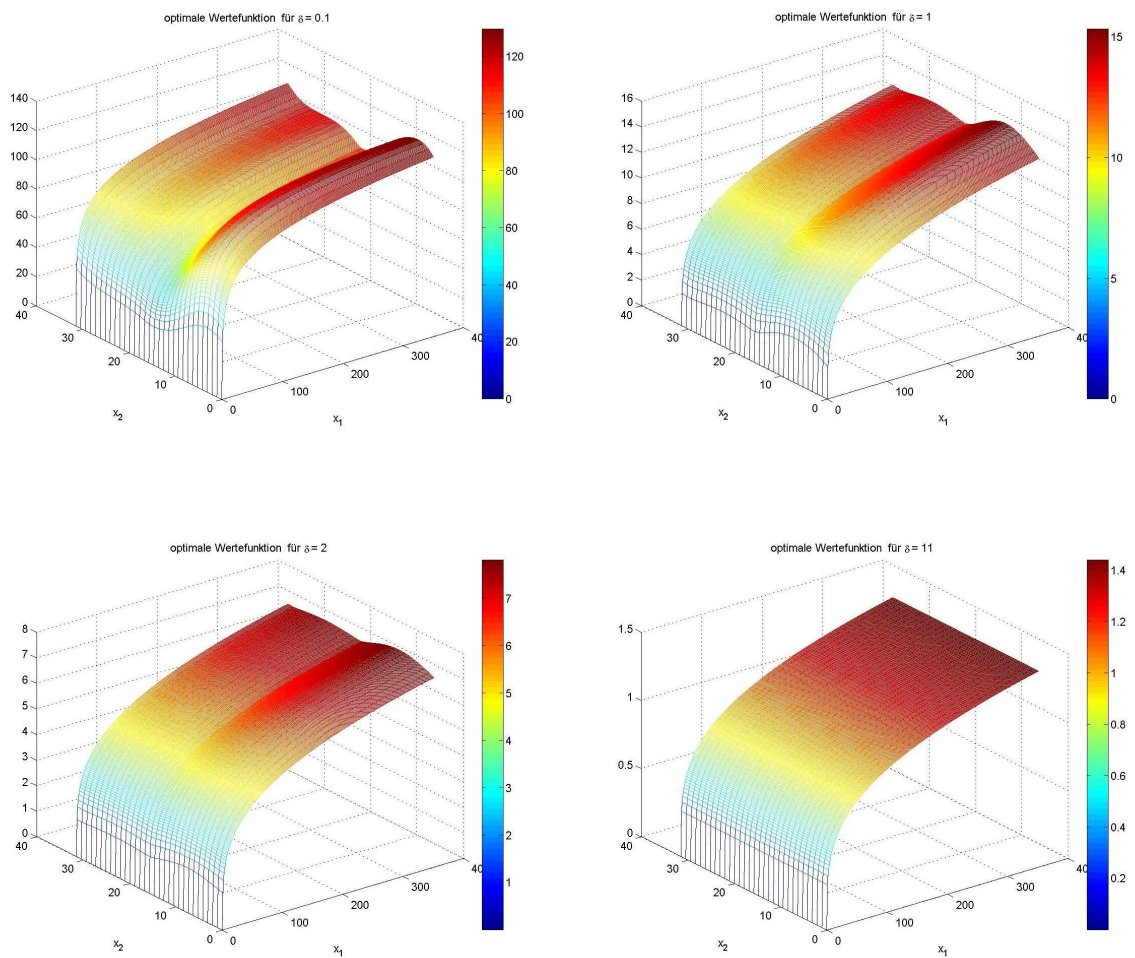
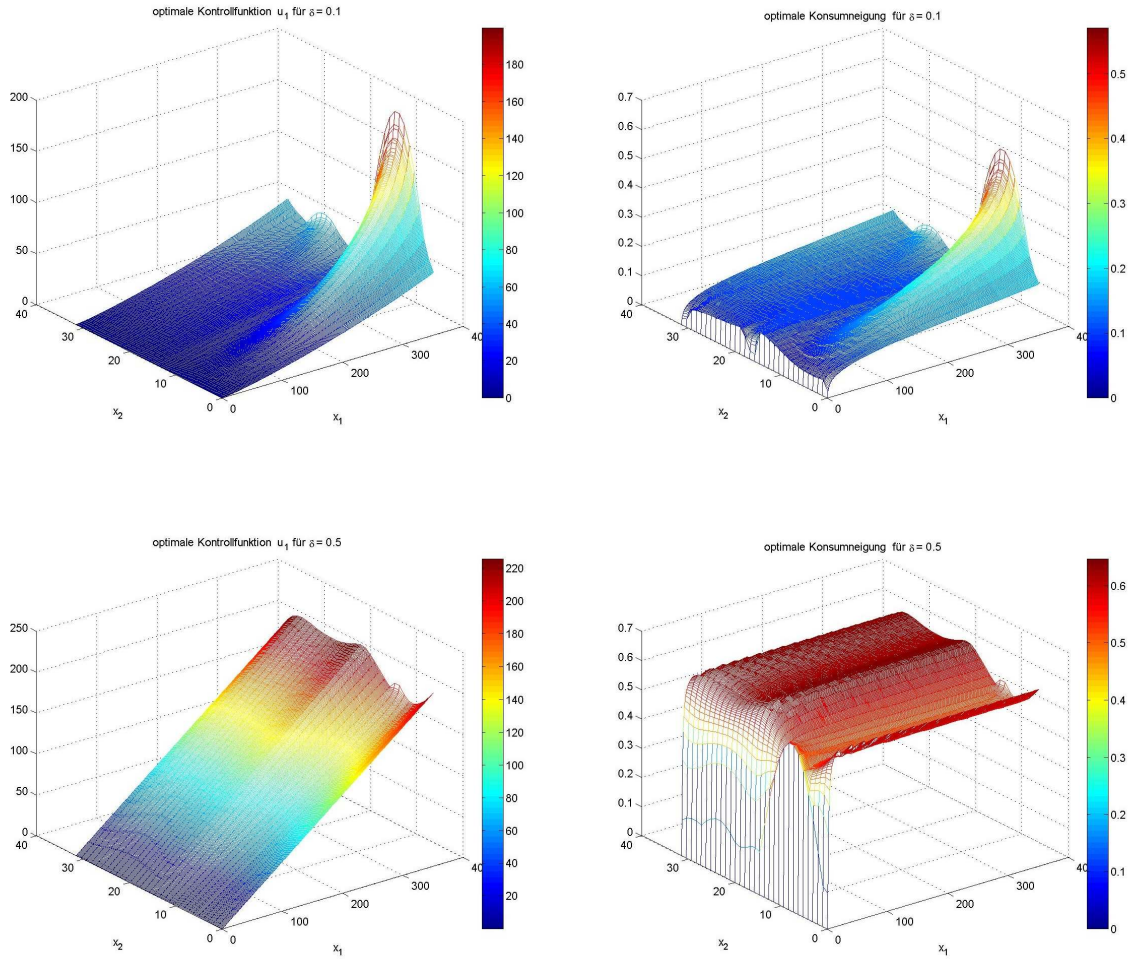


Figure 3.10: Value function for discount rates:  $\delta : 0.1$  (upper left), 1 (upper right), 2 (lower left) and 11 (lower right).

The panels in figure 3.11 show that the fluctuation of consumption and the consumption-wealth ratio shrink as the time horizon shrinks, i.e. for  $\delta \geq 1$  there are no waves any more.



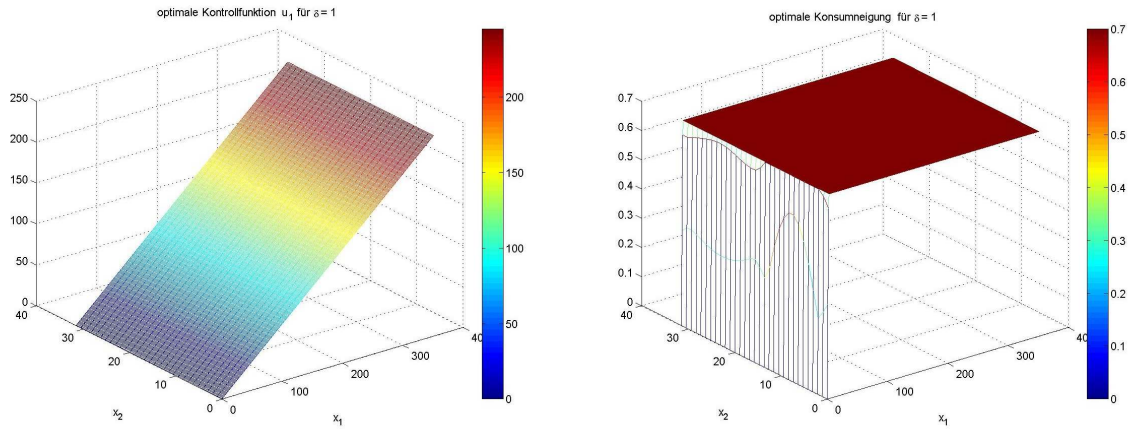
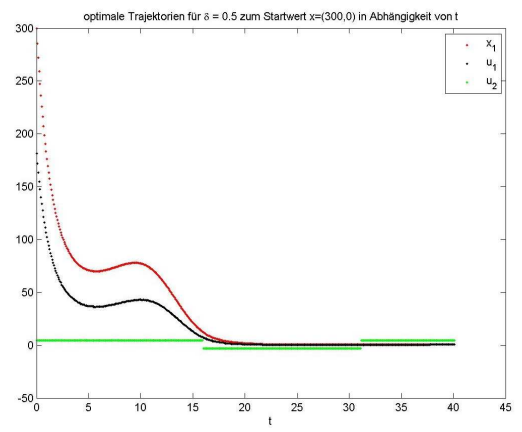
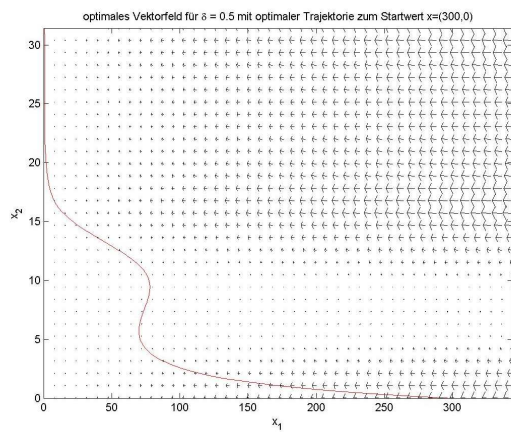
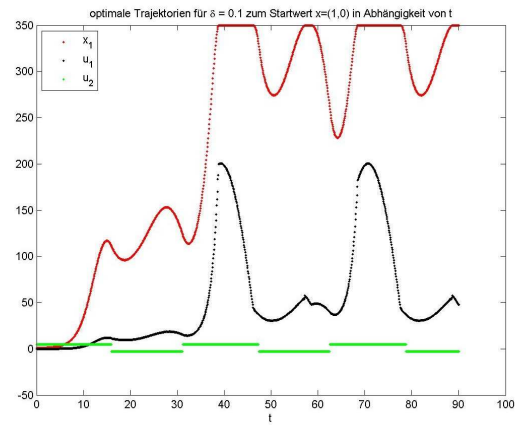
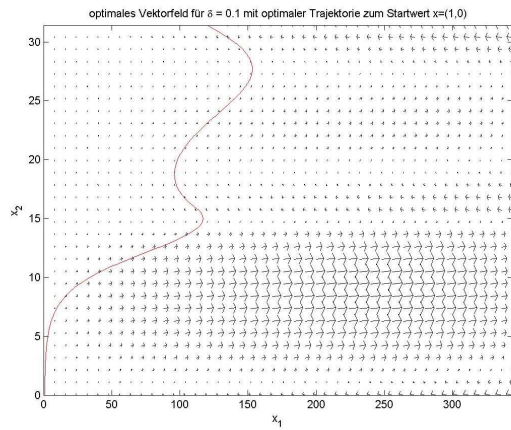


Figure 3.11: Consumption (left) and consumption-wealth ratio (right) for discount rates  $\delta$ : 0.1 (upper panel); 0.5 (middle panel); 1 (lower panel).

Figure 3.12 shows that for investment strategies over a short horizon (high  $\delta$ ) the paths for the wealth and consumption converges to zero.



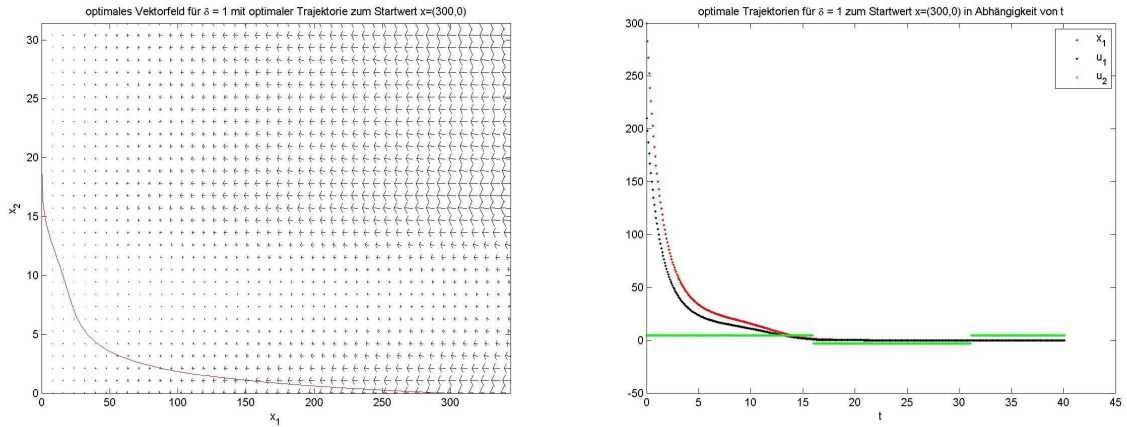


Figure 3.12: Vector fields (left) and trajectories (right) for discount rates  $\delta$ : 0.1 (upper panel), 0.5 (middle panel), 1 (lower panel).

Overall, in this section we have studied the time variation in returns, taking on a wave-like mean process. As we have shown, they create wave-like fluctuations in welfare, consumption and portfolio weights. We also could demonstrate the impact of risk aversion, the amplitude of returns and the time horizon on welfare, wealth, consumption decisions and the portfolio weights. In particular, we could observe for the time horizon it holds that welfare and the fluctuation in welfare is larger, the longer the time horizon is (the lower  $\delta$ ). On the other hand, as one would expect from theory, wealth is reduced faster as the time horizon shrinks (the lower  $\delta$ ). Moreover, as shown in all our exercises, the consumption-wealth ratio is not a constant but exhibits considerable out of steady state fluctuations. The sizable fluctuations of the consumption-wealth ratio is just a result of the optimal consumption and portfolio choice which are, through our dynamic programming algorithm, computed at any point of the state space.

## 4 A Stochastic Model with Mean Reversion in Returns

Next, we want to study a model with mean reversion in returns. We want to present an example of a stochastic consumption and portfolio choice model which has also two choice variables and two state variables. The basics of the model are analytically treated in appendix 3. Yet, the here presented slightly extended version needs to be solved numerically. We use, as stochastic processes, a process for wealth and a mean reverting interest rate<sup>15</sup> process and explore the consumption and the portfolio choice at each point of the state space. We hereby can then again obtain the consumption-wealth ratio. To avoid a third state equation we here presume a constant expected equity premium as in Campbell and Viceira (2002, ch. 3).

The model can be written for power utility as<sup>16</sup>

$$\max_{\alpha, C} \int_0^{\infty} e^{-\delta t} \frac{C^{1-\gamma}}{1-\gamma} dt \quad (15)$$

s.t.

$$dW = \{[\alpha_t(r_t + \bar{x}_t) + (1 - \alpha_t)r_t]W_t - C_t\}dt + \sigma_w dz_t \quad (16)$$

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dz_t \quad (17)$$

Denote,  $W_t$ , total wealth,  $r_t$ , the short term interest rate,  $\alpha_t$ , the fraction of wealth held as equity,  $\bar{x}$ , an expected equity premium which we assume to be a constant, yet with a stochastic shock imposed on it.<sup>17</sup>  $\theta$  is the mean interest rate and  $dz_t$  again, the increment in Brownian motion.

Following Munk et al. (2004) we assume stylized facts of the U.S. asset market such as  $\bar{x} = 0.0648$ ,  $\sigma_w = 0.0069$ ,  $\sigma_r = 0.0195$ ,  $\theta = 0.00369$ ,  $\kappa = 0.0395$ . The first and second moment reported here are annualized.

For the decision variable  $\alpha_t$  we assume  $-2 \leq \alpha \leq 2$  and for  $C_t$  we presume bounds such as  $0 < C_t < 40$ . The use a stochastic dynamic programming

<sup>15</sup>Models with mean reversion in one variable, for example, of interest rates and equity returns, are now frequently used in the portfolio modelling literature. Yet, we want to note that their empirical evidence is not as strong as multivariate factor models in forecasting returns, see Cochrane (2006).

<sup>16</sup>See Munk et al. (2004), see also Wachter (2002)

<sup>17</sup>This is similar to the assumption of Campbell and Viceira (2002, ch. 2) where they postulate "a constant risk premia and where the expected portfolio return is due entirely to variation in the riskless interest rate". Campbell and Viceira (2002:55). In our case we have, however, an additional stochastic component for the portfolio return.

algorithm that provides us with the following result for the dynamic decision paths for  $\alpha_t$  and  $C_t$ , the wealth dynamics and the consumption-wealth ratio at each point of the state space  $W - r$ . Figures 4.1-4.3 show the results of the numerical study using a stochastic variant of a dynamic programming algorithm.<sup>18</sup>

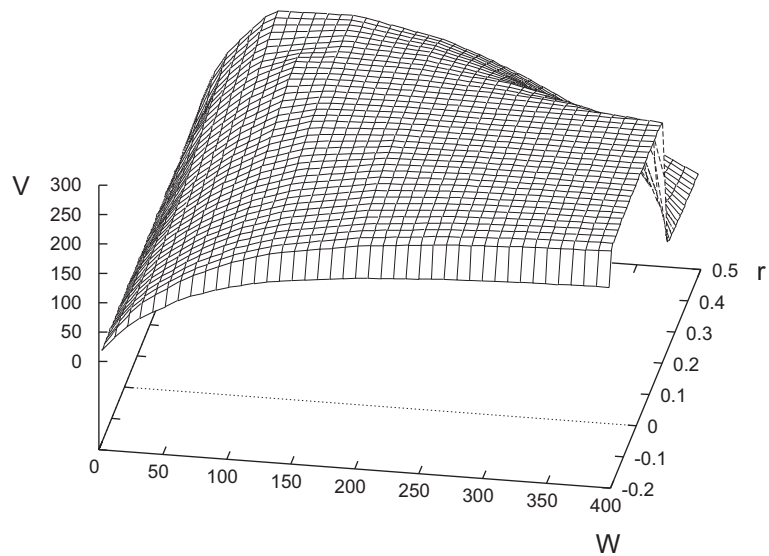


Figure 4.1: Value Function for the Wealth Dynamics

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<sup>18</sup>See Grüne and Semmler (2004).



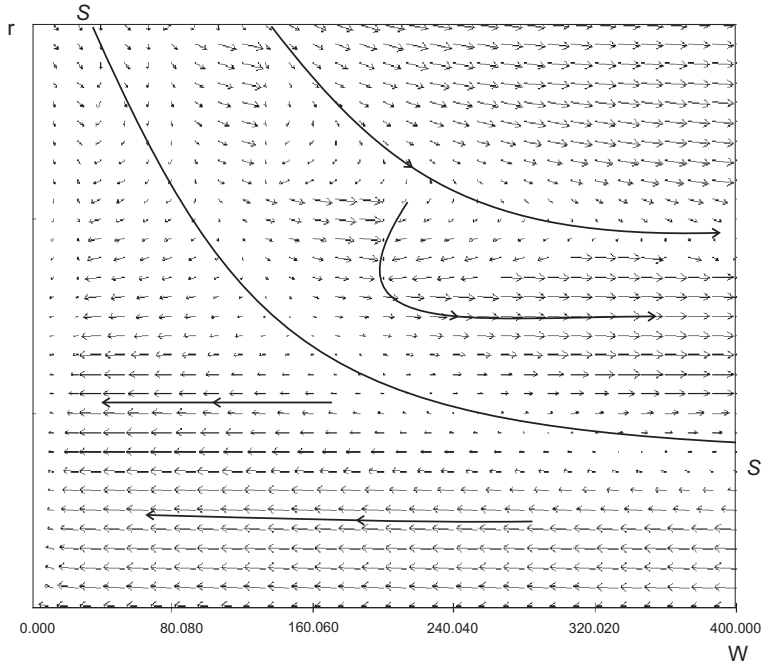


Figure 4.2: Vector field for the Wealth Dynamics

First, as observable in figure 4.1 and 4.2 there are two domains of attraction for the wealth dynamics. For low wealth (and not too high interest rates) wealth is contracting and it will finally converges to zero, as will consumption. For larger wealth and higher interest rates, given the bounded consumption  $0 < C_t < 40$ , wealth will persistently increase. This is visible from the value function, figure 4.1, and the vector field, figure 4.2. As the figures 4.1 and 4.2 suggest the two domains of attractions should be separated by a line. In some recent literature (see Grüne and Semmler, 2004) this line has been called the Skiba-line (shown in figure 4.2 by the line  $S - S$ ). Such a bifurcation can also emerge in a two dimensional control problem similarly of what we have observed in section 2 for the one-dimensional problem.

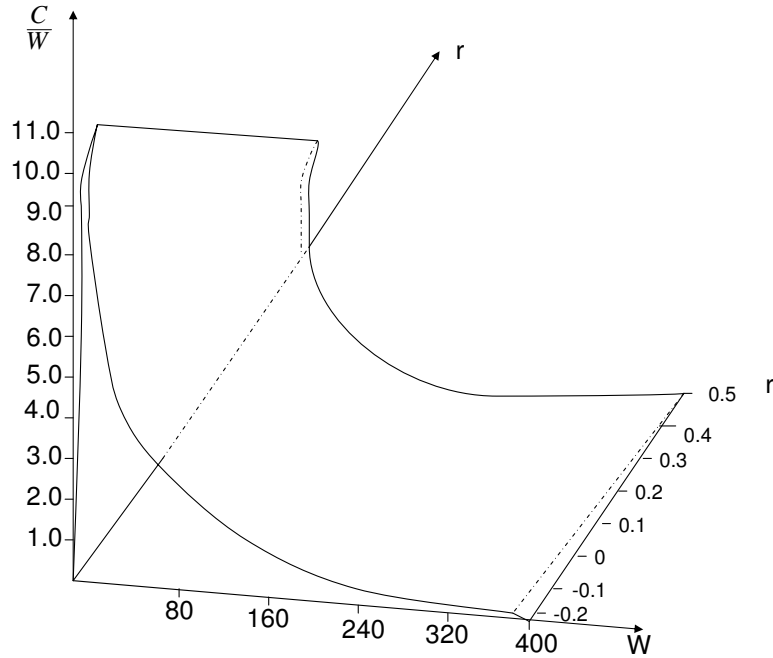


Figure 4.3: The consumption-wealth ratio (depending on the state space  $W - r$ )

Second, we want to note, as our numerical results show, that the consumption-wealth ratio strongly varies in the state space  $W - r$ . In particular, as figure 4.3 shows, at different levels of wealth,  $W$ , the consumption-wealth ratio shows large differences. Since in our model of eqs. (15)-(17) the expected return on the risky asset is just an increasing function of the risk free rate,  $r$ , the model does not seem to imply that the consumption-wealth ratio is a good predictor of future returns, as for example Lettau and Ludvigson (2001) state.<sup>19</sup>

<sup>19</sup>This at least holds if consumption, as in our model, is bounded from above. On the other hand, as shown in sections 2 and 3 the discount rate, risk aversion and time horizon are important as well for the consumption-wealth ratio.

## 5 Conclusions

In this paper we have followed up some recent research in dynamic consumption and portfolio choice theory. We first start with a model with a constant return and then extend it where consumption and portfolio decisions are made when returns are time varying. In section 3 of the paper we have chosen a framework where our decision maker can choose the optimal consumption path without constraint, but the choice of portfolio weights was constrained within bounds. Since we have not introduced transaction costs for changing portfolio weights, our constraints for the portfolio weights may be a reasonable procedure to avoid unreasonably large positive and negative weights.

We could show that when there are low frequency movements in the returns, and thus time varying investment opportunities, a buy and hold strategy is not reasonable, but rebalancing of the portfolio is needed in order to increase wealth and welfare. Readjustments of consumption and rebalancing of the portfolio should, as we have argued, follow the low frequency component of the returns from the financial assets.

In contrast to Campbell and Viceira (1999, 2002, chs. 2-4) in all of our model variants the consumption-wealth ratio is not approximated but accurately computed at each point in the state space. In Campbell and Viceira (1999, 2002) the consumption-wealth ratio is approximated, and their solution becomes less accurate as the variability of the ratio increases (see Campbell and Viceira 1999:442). Yet, as we have shown, the optimal solution of the consumption-wealth ratio can greatly vary across the state space if the optimal consumption and portfolio decisions are correctly computed<sup>20</sup> which is not a problem for our procedure since it gives us global solutions with sufficient accuracy.

We also follow up the research on mean reversion processes. We study a stochastic case with two decision variables (consumption and portfolio weights) and two state variables, with wealth and interest rate as state variables where the interest rate follows a mean reverting process. Here, we presume consumption to be bounded (in particular to have an upper bound). In this model the expected equity premium is a constant and the expected equity return moves with the interest rate.<sup>21</sup> Here too, we could demonstrate that the consumption-wealth ratio can greatly vary. We also could observe again, as in the model of section 2, the possibility of a bifurcation in the dynamics. Future research should address the issue of dynamic consumption

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<sup>20</sup>The approximate solution of Campbell and Viceira (1999) holds only for a low parameter of risk aversion, and for a constant risk-free rate.

<sup>21</sup>Note that this model version in its set up is similar to Campbell and Viceira (1999).

and portfolio decisions with time varying asset returns more extensively in the context of a stochastic model.<sup>22</sup>

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<sup>22</sup>For a further use of dynamic programming to solve for dynamic consumption and portfolio decisions in a stochastic framework, see Chiarella et al. (2007).

# Appendix 1: Sketch of the Dynamic Programming Algorithm

We here briefly describe the dynamic programming algorithm<sup>23</sup> as applied in Grüne and Semmler (2004) that enables us to numerically solve the dynamic model as proposed in section 4. The feature of the dynamic programming algorithm is an adaptive discretization of the state space which leads to high numerical accuracy with moderate use of memory.

Such algorithm is applied to discounted infinite horizon optimal control problems of the type introduced in section 4. In our model variants we have to numerically compute  $V(x)$  for

$$V(x) = \max_u \int_0^\infty e^{-\delta t} f(x, u) dt$$

s.t.  $\dot{x} = g(x, u)$

where  $u$  represents the control variable and  $x$  a vector of state variables.

In the first step, the continuous time optimal control problem has to be replaced by a first order discrete time approximation given by

$$V_h(x) = \max_j J_h(x, u), \quad J_h(x, u) = h \sum_{i=0}^{\infty} (1 - \theta h) U f(x_h(i), u_i) \quad (\text{A1})$$

where  $x_u$  is defined by the discrete dynamics

$$x_h(0) = x, \quad x_h(i+1) = x_h(i) + hg(x_i, u_i) \quad (\text{A2})$$

and  $h > 0$  is the discretization time step. Note that  $j = (j_i)_{i \in \mathbb{N}_0}$  here denotes a discrete control sequence.

The optimal value function is the unique solution of a discrete Hamilton-Jacobi-Bellman equation such as

$$V_h(x) = \max_j \{hf(x, u_o) + (1 + \theta h)V_h(x_h(1))\} \quad (\text{A3})$$

where  $x_h(1)$  denotes the discrete solution corresponding to the control and initial value  $x$  after one time step  $h$ . Abbreviating

$$T_h(V_h)(x) = \max_j \{hf(x, u_o) + (1 - \theta h)V_h(x_h(1))\} \quad (\text{A4})$$

---

<sup>23</sup>This algorithm builds on Bellman (1967) and Bardi et al. (1997).

the second step of the algorithm now approximates the solution on grid  $\Gamma$  covering a compact subset of the state space, i.e. a compact interval  $[0, K]$  in our setup. Denoting the nodes of  $\Gamma$  by  $x^i, i = 1, \dots, P$ , we are now looking for an approximation  $V_h^\Gamma$  satisfying

$$V_h^\Gamma(X^i) = T_h(V_h^\Gamma)(X^i) \tag{A5}$$

for each node  $x^i$  of the grid, where the value of  $V_h^\Gamma$  for points  $x$  which are not grid points (these are needed for the evaluation of  $T_h$ ) is determined by linear interpolation. We refer to the paper cited above for the description of iterative methods for the solution of (A5). Note that an approximately optimal control law (in feedback form for the discrete dynamics) can be obtained from this approximation by taking the value  $j^*(x) = j$  for  $j$  realizing the maximum in (A3), where  $V_h$  is replaced by  $V_h^\Gamma$ . This procedure in particular allows the numerical computation of approximately optimal trajectories.

In order to distribute the nodes of the grid efficiently, we make use of a posteriori error estimation. For each cell  $C_l$  of the grid  $\Gamma$  we compute

$$\eta_l := \max_{k \in C_l} | T_h(V_h^\Gamma)(k) - V_h^\Gamma(k) |$$

More precisely we approximate this value by evaluating the right hand side in a number of test points. It can be shown that the error estimators  $\eta_l$  give upper and lower bounds for the real error (i.e., the difference between  $V_j$  and  $V_h^\Gamma$ ) and hence serve as an indicator for a possible local refinement of the grid  $\Gamma$ . It should be noted that this adaptive refinement of the grid is very effective for computing steep value functions and models with multiple equilibria, see Grüne and Semmler (2004).

## Appendix 2: Low Frequency Components of Returns

We want to note that much work has been undertaken to find appropriate forecasting variables for time varying asset returns. The most popular ones since the 1980s, are the dividend-price ratio, dividend yield or the consumption-wealth ratio<sup>24</sup> which were much celebrated to have good forecasting power for time varying asset returns.<sup>25</sup> In other recent literature

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<sup>24</sup>For the former, see Campbell and Viceira (2002) and for the latter, see Lettau and Ludvigson (2001).

<sup>25</sup>See the literature cited in footnote 4.

univariate mean reversion processes were used to predict time varying expected returns. See Campbell and Viceira (2002), Munk et al. (2004) and Wachter (2002) for their justification and use in dynamic portfolio decision models, and see Cochrane (2006) for a critical view on the use of univariate mean reversion processes.

Other empirical literature point to the superior performance of the use of moving average returns, i.e., sample mean, as forecasting variable in particular for out-of-sample predictions. The moving average technique, which captures the low frequency component of asset returns, is also often used by market practitioners for out-of-sample forecast with quite superior performance as compared to other forecasting methods.<sup>26</sup> Indeed, as shown in Lettau and Nieuwerburgh (2006) many of the financial ratios exhibit a shifting mean, resulting in uncorrect or spurious regressions. We thus here simply use time varying mean returns and represent them by low frequency components of actual asset returns. Those are employed in the model of section 3.

The long swings that are visible in the low frequency components of asset returns, see figures A1-A3 below, are also supported by recent theoretical research on asset pricing using loss aversion theory. In Barberis et al. (2001) and Grüne and Semmler (2005) it is shown that including asset gains and losses in investors' preferences gives rise to low frequency movements in asset returns and to mean reverting processes. An econometric study that uses a regime change model gives empirical support for such an approach, see Zhang and Semmler (2006).

The data on the returns, for the time period 1929-2000, are from Bekaert et al. (2006). The subsequent appendix presents filtered time series data using the HP filter<sup>27</sup> on equity return (figure A1), short term interest rate (figure A2) and bond return (figure A3).

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<sup>26</sup>Goyal and Welch for example state "Neither the dividend-yield nor the dividend price ratio had both in-sample and out-of-sample performance that should one lead to believe that it could out-perform the simple prevailing equity premium average in an economically and statistically significant manner" (Goyal and Welch, 2003: 653).

<sup>27</sup>We use here the HP filter, since we would like to filter out low frequency components of asset returns at business cycle frequency. The use of the BP-filter gave us similar results. Rick Ashley has guided us to several filters of different types, that seem to be applicable for our purpose, see Ashley's web-site at Virginia Tech.

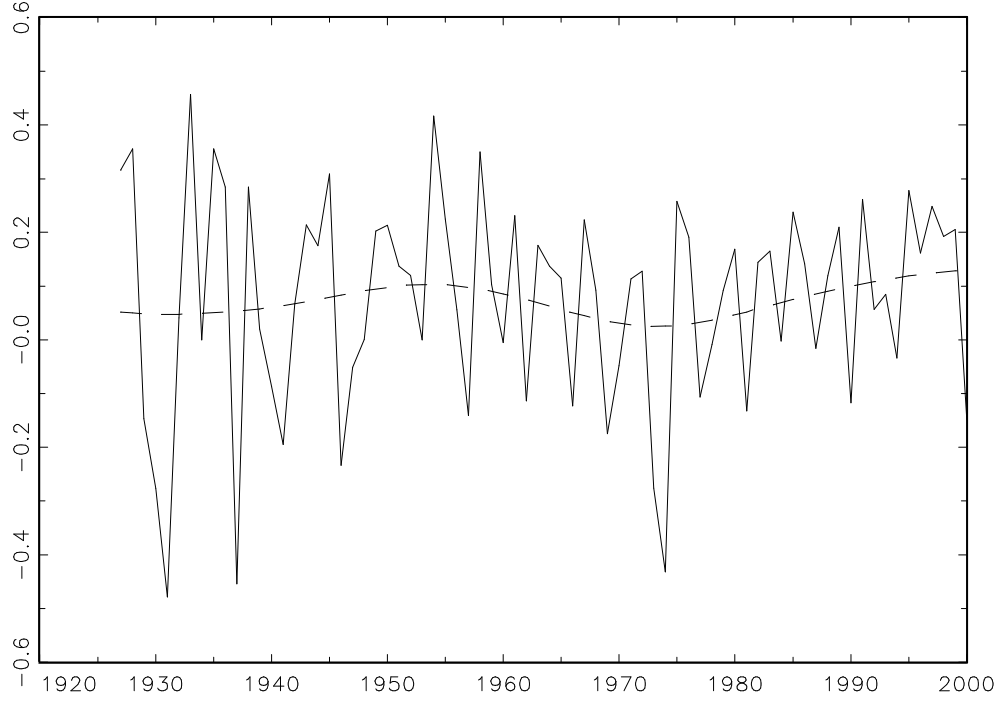


Figure A1: Actual and low frequency component of real equity return



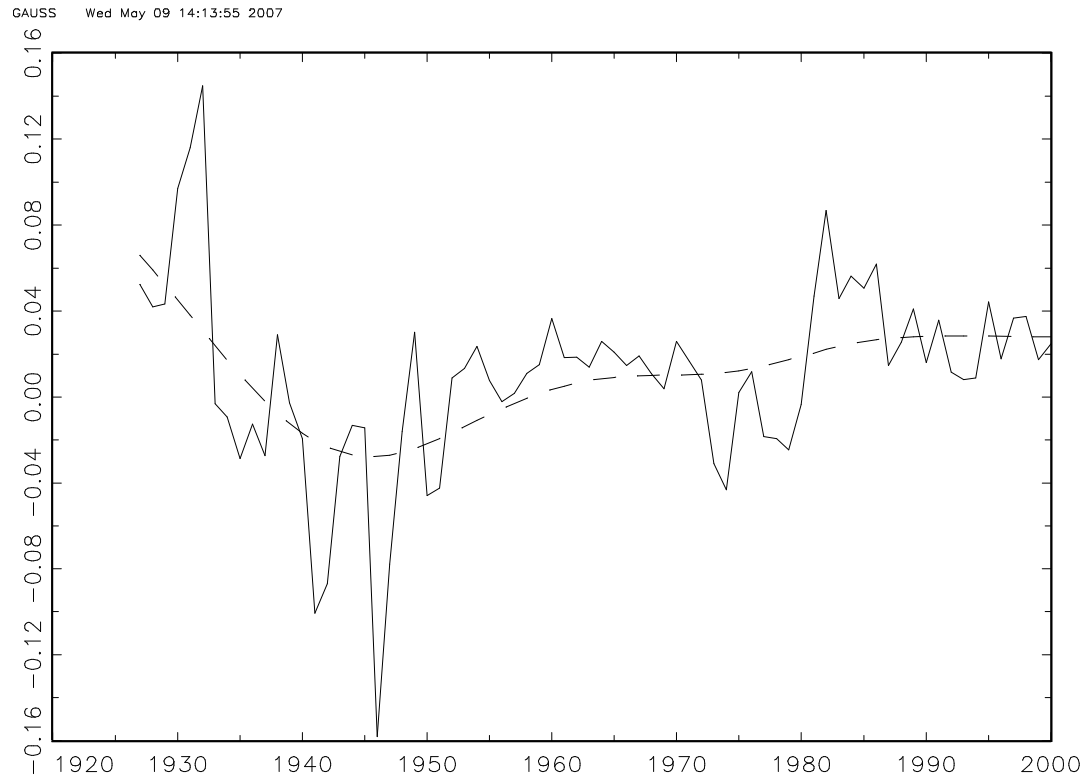


Figure A2: Actual and low frequency component of short-term real interest rate

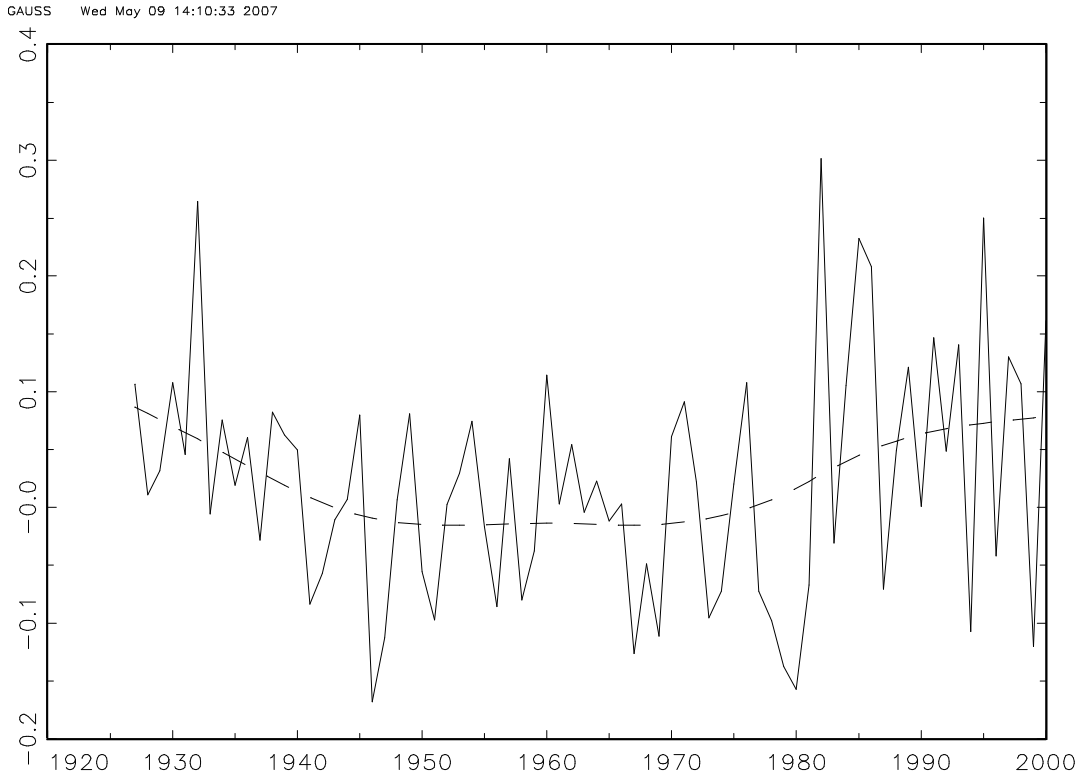


Figure A3: Actual and low frequency component of real bond return

### Appendix 3: Stochastic Version of the Model

We briefly consider here dynamic portfolio decisions in stochastic environment. We study a model with two assets and two returns. It represents the analytical form of the stochastic version solved in section 4. Yet, here we work with one asset, delivering a risk-free rate, with a constant return, and the other asset, a risky asset, generating a risky return. This can be reduced to a dynamic decision problem with one state and one control variable.<sup>28</sup>

<sup>28</sup>Kamien and Schwartz (2001, sect. 22)

The optimal expected return from the dynamic decision can be expressed in current value terms, in general form as

$$V(x_0) = \max E \int_0^\infty e^{-\delta t} f(x, u) dt \quad (18)$$

$$\text{s.t. } dx = g(x, u)dt + \sigma(x, u)dz, \quad x(t_0) = x_0, \quad (19)$$

This represents a stochastic decision problem with  $x$  as state variable,  $u$  as control variable and a Brownian motion in equ. (19). From (18) and (19) one can obtain a HJB-equation in stochastic form

$$\delta V(x) = \max_u (f(x, u) + V'(x)g(x, u) + (1/2)\sigma^2(x, u)V''(x)) \quad (20)$$

Let us turn the above general form into a simple example, see Merton (1973, 1990). The example, we want to study, has two controls and one state variable representing a model of allocating wealth among current consumption, investment in a risk-free asset, and investment in a risky asset. We here too exclude transaction costs. Agents are not restricted in choosing portfolio weights. Denote  $W$ , total wealth,  $\alpha$ , fraction of wealth in the risky asset,  $R_f$ , return on the risk-free asset,  $R_e$ , expected return on the risky asset,  $R_e > R_f$ ,  $\sigma^2$ , variance per unit time of the return on the risky asset, and  $C$ , consumption. Presume preferences  $U(C)$ ,  $C^b/b$  with  $b = 1 - \gamma$ .

The change of wealth can be denoted by

$$dW = [(1 - \alpha)R_fW + \alpha R_eW - C]dt + \alpha W \sigma dz. \quad (21)$$

There is a deterministic fraction of wealth which is determined by the return on the funds in the risk-free asset, plus the expected return on the funds in the risky asset, minus consumption.

The stochastic term at the end of equ. (18) denotes the increment of a Brownian motion, representing the stochastic part of the risky return. The aim of the holder of wealth is the maximization of an expected discounted utility flow. We again assume a model with an infinite horizon:

$$\max_{C, \alpha} E \int_0^\infty (e^{-\delta t} C^b/b) dt \quad (22)$$

$$\text{s.t. (21) and } W(0) = W_0.$$

This infinite horizon decision problem has indeed one state variable  $W$  and two control variables  $C$  and  $\alpha$ . The model of sect. 2.1 represents a problem with just one state variable and one choice variable. Using the specifications of (21) and (22), (20) we can write the HJB-equation in stochastic form.

$$\delta V(W) = \max_{C, \alpha} (C^b/b + V'(W))[(1-\alpha)R_f W + \alpha R_e W - C] + (1/2)\alpha^2 W^2 \sigma^2 V''(W). \quad (23)$$

Some calculus<sup>29</sup> provides us with the maximizing values of  $C$  and  $\alpha$  for the given parameters of the problem, the state variable  $W$ , and the unknown function  $V$ :

$$C = [V'(W)]^{b/(b-1)}, \quad \alpha = V'(W)(R_f - R_e)/\sigma^2 W V''(W). \quad (24)$$

It is assumed that the optimal solution involves investment in both assets for all  $t$ . Using (24) and (23) and simplifying we obtain

$$\delta V(W) = (V')^{b/b-1}(1-b)/b + R_f W V' - (R_f - R_e)^2 (V')^2 / 2\sigma^2 V''. \quad (25)$$

One can try a solution to this nonlinear second order differential equation of the form

$$V(W) = AW^b, \quad (26)$$

Hereby  $A$  is a positive parameter to be determined. One can compute the required derivatives of (26) and use the results in (25). With some simplification, one obtains

$$Ab = \{[\delta - R_f b - (R_f - R_e)^2 b / 2\sigma^2 (1-b)] / (1-b)\}^{b-1}. \quad (27)$$

Thus, the optimal current value function is (26), with  $A$  as specified in (27). In order to find the optimal choice  $C$ , use equs. (26) and (27) in equ. (24):

$$C = W(Ab)^{1/(b-1)}, \quad \alpha = (R_e - R_f) / (1-b)\sigma^2. \quad (28)$$

This means that the household chooses a constant consumption wealth ratio at each instant of time only if the equity premium remains a constant. The optimal choice depends on the parameters. It varies with the discount rate and with the variance of the risky asset. Similarly to a static case, the optimal wealth chosen for the two kinds of assets is a constant, independent of total wealth, as long as the equity premium and variance  $\sigma^2$  remain constant. Yet, the portion devoted to the risky asset varies with the equity premium. It is related to the variance of that return and the risk aversion parameter,  $\gamma$ , since  $b = 1 - \gamma$ . The above model is usually used for dynamic portfolio choice problem with stochastic equity return and constant risk-free interest rate. Although it has two control variables it has only one state variable.

<sup>29</sup>For details, see Kamien and Schwartz (2001, sect. 22)

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