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History Dependence and Global Dynamics in Models with Multiple Equilibria

by

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PRELIMINARY VERSION

Abstract

A wealth of literature, reviewed in the first section of this paper, is concerned with the occurrence of multiple equilibria in economic optimization models and with the resulting history dependence of optimal solutions. Typically, the existence of multiple equilibria is associated with convex-concave production functions, externalities, market imperfections, expectational phenomena, and the like. Less known is that (1) this phenomenon is also possible in intertemporal concave optimization model; (2) the threshold separating the optimal trajectories towards the one or the other long-run optimal outcome does not necessary coincide with an unstable steady-state; (3) unstable steady-states may generically be non-optimal; (4) the policy function at the thresholds is frequently not continuous; and (5) local stability analysis may yield information on the occurrence of these properties. In the paper, we demonstrate and discuss these properties in the case of a one-dimensional state space, with an extension to the two-dimensional case. Since in most cases, the first four properties (1)-(4) cannot be addressed analytically, we present three numerical methods for their investigation.

Keywords: Dynamic optimization models, Multiple equilibria, History dependence, Skiba thresholds, Numerical methods.

JEL classification: C61, C62

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1 Introduction

Recent work on dynamic economics has stressed the fact that economic outcomes may be history dependent. The term history dependence has been made popular by the work of Bryan Arthur describing economic paths of new technologies that exhibit increasing returns and positive feedbacks. Such increasing returns and positive feedbacks may arise globally or locally only. Locally increasing returns, for example, can already be found in early models in development economics using first convex and then concave production functions. From the onset it has been recognized that such convex-concave production functions may imply multiple steady-states – usually, two saddle-point outer and one unstable middle steady-state. As a consequence, the long-term behavior of the economy will be history dependent: According to the conditions prevailing in the first phases of development, the economy will converge to the one or the other saddle-point. Consequently, there exists a threshold where the dynamics leading to these two different long term solutions separate. Following the pioneering article of Skiba (1978), such thresholds have been occasionally called Skiba points in the economic literature. We equivalently use both denominations.

In the 1980s, dynamic models with multiple steady-states and history dependence have been proposed in numerous areas in economics. Such models can be found in development economics, in trade and resource allocation, in labor market search and matching theory, in the economic theory of addiction, in endogenous growth theory, in resource and environmental management problems, in models of monetary policy, in regulatory economics, and in game theory. Those are reviewed in Section 2 of this paper.

As this review will show, the roads leading to multiple steady-states and history dependence are numerous. In the rest of the paper, we concentrate on the class of models where the emergence of multiple steady-states may arguably be considered the most unlikely, namely, on representative agent dynamic optimization models under perfect foresight. Already at this point, it may be useful to give an intuitive flavor of the relationship between history dependence, the existence of multiple steady-states, and the existence of a threshold in such models. First of all, note that history dependence is given if and only if, depending on the initial conditions, the optimal solutions of the dynamic system of interest converge towards two or more distinct attractors. These attractors can be saddle-points, stable steady-states, or other points such as the origin if the feasible state-space is bounded by non-negativity conditions, or such as + infinity if the optimal trajectories grow without bounds. With some abuse of language, all these attractors will be termed stable steady-states in this paper. Second, the existence of two or more stable steady-states implies the existence of at least one unstable steady-state – this time in the strict sense of an unstable stationary solution. Third, the existence of optimal trajectories towards the one or the other attractor implies the existence of a optimal (set of) point(s) on which one is indifferent between converging towards to one or the other stable steady-state, that is, of a Skiba point (set) or threshold.

Thus motivated, the paper gives a synthetic presentation of the properties necessary for the emergence of multiple steady-states (meaning at least two attractors and one unstable steady-state) in efficient dynamic optimization models. Contrary to a common belief, it is shown that history dependence is possible even in strictly concave models. Furthermore, the paper discusses in detail another little known property. If strict concavity is not given, the Skiba points generically do not coincide with the unstable steady-states, and the latter are not necessarily optimal. Local eigenvalue analysis can give information on whether one can expect this to be the case.

Most of the analysis is restricted to one-dimensional problems. However, the paper also addresses the occurrence of multiple steady-states and thresholds in two-dimensional dynamic optimization problems. These thresholds, or Skiba curves, are a straightforward but non-trivial generalization to two-dimensional spaces of the one-dimensional Skiba points. The optimal actions of the agents are indeterminate on any point on the Skiba curve, since on this curve the optimal pay-off (the value function) is the same whether one tends towards the one or the other steady-state.

When the Skiba points – or Skiba curves – do not coincide with unstable steady-states, it is in general impossible to determinate them analytically. Numerical procedures are required. The paper proposes three different numerical procedures to find Skiba points while gaining insight on the global dynamics of the underlying economic problem. Those methods build on the Hamilton-Jacobi-Bellman (HJB) equation, on Pontryagin´s maximum principle and the associated Hamiltonian, and on dynamic programming. Their respective strenghts and weaknesses in detecting thresholds are discussed.

The paper is organized as follows. Section 2 gives a short survey of the literature on multiple steady-states and history dependent outcomes. Section 3 introduces the basic framework, that is, the class of dynamic optimization problems considered. Section 4 presents and discusses different necessary conditions for history dependence. Section 5 is devoted to the presentation of numerical methods for finding the Skiba points. Section 6 concludes the paper. The appendix demonstrates the usefulness of the HJB-equation for computing the value function and thresholds.

2 Dynamic optimization models with multiple steady-states

Numerous examples of dynamic economic models with multiple steady-states (that is, with at least two optimal attractors and one unstable steady-state) and history dependence can be found in the economic literature.

1. **Convex-concave production function**. One of the simplest is given by one-dimensional models of capital accumulation with a convex-concave production function to be found, among others, in the literature on development economics. There, a convex-concave production function may arise due to social inputs such as institutions or human capital. A Skiba point separates paths leading to stable high and low income steady-states. Thus, these models can explain the co-existence of countries with low and of countries with high per capita income as a function of their respective initial conditions alone. A widely used version of convex-concave production function can be found in Skiba (1978) and Azariadis and Drazen (1990).[1] Similarly, multiple steady-states can also arise in a one-capital-good model if nonlinear adjustment costs of investment are assumed. This type of model can be found in Blanchard (1983) and is shown to give rise to multiple steady-states in Semmler and Sieveking (1999a). Related to this is the observation of Arthur (1989, 1994) that increasing returns to scale lead to outcomes highly sensitive to initial conditions and thus path dependent.

2. **Expectation formation**. Other roads can lead to multiple steady-states in one-state-variable dynamic models. Among them are expectational mechanisms. Krugman (1991) considers for example a two-sectors economy where the first sector exhibits constant, the second increasing returns to scale with respect to the only input, labor. The marginal productivity of labor and thus the wage rate is higher (lower) in the first sector than in the second depending on whether most of the labor force is employed in the first or the second sector. Therefore, it is rational for a worker to move to the sector to which he or she expects that the others are going to move. The model has three steady-states, an unstable middle one, and two saddle-point ones where all workers have moved to either the first or the second sector. The emergence of multiple outcomes through an expectations formation mechanism can be also found in Arthur's (1994) El-Farrol problem.

[1]Econometric tests of such models with threshold effects are undertaken in Durlauf and Johnson (1995) and Bernard and Durlauf (1998).

3. **Search and matching**. Further examples can be found in labor market search and matching theory. There, the co-existence of persistent high or low rates of unemployment can be derived from a microeconomic foundation of labor market search theory along the lines of Diamond (1982). Thus, in Mortensen (1989) and Howitt and McAffee (1992), nonlinearities in firms' pay-offs stem from a combination of labor market search cost and net pay-offs arising from employing an additional unit of labor. In the Mortensen (1989) model the search cost is constant. By contrast, in Howitt and McAffee (1992) the search cost depends upon unemployment and job vacancies and is nonlinear. In either case, multiple unemployment steady-states can arise.

4. **Monetary policy models**. Some recent monetary policy models exhibit similar properties, see Benhabib, Schmitt-Grohe and Uribe (1998). In this model, consumers' welfare is affected positively by consumption and cash balances and negatively by the labor effort and an inflation gap from some target rates. The model admits unstable as well as stable high level and low level inflation rate steady-states. Moreover, there can be indeterminacy in the sense that any initial condition in the neighborhood of one of the unstable steady-states is associated to an optimal path. The same kind of dynamics can also be found in Greiner and Semmler (1999).

5. **Models of addiction**. Multiple steady-states play also an important role in models of addiction, see Orphanides and Zervos (1995, 1998). Here, three steady-states exists, the middle one being unstable. As a consequence of small shocks in drug consumption or of the impact of enforcement policies, the system can converge towards either the low level steady-state where there is no drug consumption, or towards the high level steady-state with addiction. The threshold separating the optimal trajectories leading to either the low or the high level steady-state does not necessarily coincide with the middle unstable steady-state.

6. **Endogenous growth**. Recent literature on economic growth considers endogenous growth models with two types of capital goods, either physical capital and human capital along the line of Lucas (1988) or physical capital and knowledge capital following Romer (1990). Multiple steady-states can arise in both cases. For extended Lucas models, they are shown to exist in Chamley (1993), Benhabib and Perli (1994), Xie (1994) and Ladron-de-Guevara, Ortigueira and Santos (1997). These authors obtain the multiple steady-states either by introducing externalities in the production for human capital, or by

including leisure in the utility function of the representative agent. Extensions of the Romer (1990) model are used in Benhabib and Perli (1994) and Evans, Honkapohja and Romer (1999) to study multiple steady-states arising from complementarities between inputs. See also Matsuyama (1991) and Santos (1999). Likewise, multiple steady-states can arise in a one-capital-good model where the adjustment cost is a nonlinear function of the change of investment as, for example, in Haunschmied, Kort, Hartl and Feichtinger (2000). This model, however, is a two-state-variables model. This significantly complicates the determination of the threshold (which is given by a Skiba curve instead of a Skiba point).

7. **Renewable resources**. Dynamic models of renewable resources with two state variables can easily exhibit multiple steady-states. In the two-resources model of Sieveking and Semmler (1997), the resource dynamics may exhibit at least three steady-states depending on the type of interaction between the resources — competitive, predator-prey or cooperative. Again, the middle steady-state is unstable, while the outer two are saddle-points. Similarly, multiple steady-states have been shown to exist in ecological management problems, see Lewis and Schmalensee (1983), Tahvonen and Salo (1996), and Tahvonen and Withagen (1996). Recently, Brock and Starret (1999), Dechert and Brock (1999), Mäler (2000), Mäler, Xepapadeas and de Zeeuw (2000) studied a lake management problem where multiple steady-states arise due to non-concavity. The middle steady-state is unstable. At the stable low level steady-state, the lake's self-regenerative forces are strong enough to keep it clean. The high level stable steady-state corresponds to a situation where the lake has flipped over.

8. **Regulatory economics**. Another category of dynamic convex-concave models with history dependent outcomes can be found in regulatory economics. Brock and Dechert (1985) investigate dynamic Ramsey pricing, with the interesting result that the maintenance of a public service (say, of a railroad network) depends on the initial conditions (rails won't make it anymore in Australia and Africa but will be kept in Europe). Furthermore, the viability of a service does not imply that a private firm will maintain the service. Profit maximization may imply liquidation in the long run, although a viable and stable steady-state exists. All these properties are attributed to convex-concave functions, with the convexity resulting from locally increasing returns. Along similar lines, Dechert (1984) considers the familiar Averch-

Johnson effect within a dynamic framework using a convex-concave production function. Brock (1983) investigates a positive problem, lobbying, within a convex-concave setting. These models all support the by now familiar result of multiple steady-states, Skiba points, etc.

9. **Differential games**. In differential games, multiple feedback Nash equilibrium steady-states can arise even in the case of a linear-quadratic game with one state variable. This results from the fact that the differential equation implied by the Hamilton-Jacobi-Bellman equation, that characterizes the optimal feedback strategies, lacks a boundary condition. The multiplicity of equilibria was first noted in Tsutsui and Mino (1990). In an early application, Dockner and Long (1993) argue that a proper choice of nonlinear strategies can resolve in a non-cooperative way the tragedy of the commons. Less known is the existence of multiple open-loop Nash equilibria, including the possibility of limit cycles, see Wirl, Feichtinger, Dawid and Novak (1997). These equilibria form a continuous family, to which another, typically unstable steady-state may be associated. Thus, differential games allow, in nonlinear strategies in linear quadratic games, for an entire family of solutions. However, there is no history dependence if one requires the strategies to be stable.

This survey has confirmed a generally acknowledged fact: Multiple steady-states, thresholds, and history dependence are fairly ubiquitous in dynamic economic models. Numerous mechanisms may generate them. In the remainder of this paper, we exclude the arguably most straightforward possibilities for generating history dependence – externalities, expectation formation, coordination failures, ... – by restricting our discourse to the emergence of unstable steady-states and history dependence in representative agent intertemporal optimizations models under perfect foresight. Two little known and, possibly, counter-intuitive points will emerge from the analysis. Multiple steady-states and history dependence is possible also in strictly concave models. The unstable steady-states are not necessarily optimal, and do not always coincide with the thresholds separating the domains of attraction between stable steady-states.

3 Framework and optimality conditions

We consider in this paper inter-temporal optimization problems $P(\mathbf{a})$ of the type:

$$V(\mathbf{a}) \equiv \max_{u(t)\in U} \int_0^\infty e^{-rt} f_0(x(t), u(t))dt, \tag{1}$$

$$s.t.\ \dot{x} = f(x(t), u(t)),\ x(0) = \mathbf{a}, \tag{2}$$

where x is the state, u the control, U a compact set of admissible controls, t the time index, and $r > 0$ a discount factor. Furthermore, $f_0(x(t), u(t))$ is a return function, $f(x(t), u(t))$ is a transition function that describes the state dynamics, and $V(\mathbf{a})$ is the value function, i.e. the maximum aggregate present value of benefits when starting at $x(0) = \mathbf{a}$. The problem is parameterized in terms of $\mathbf{a}$ since the initial conditions are a crucial ingredient of a history dependent outcome.

In line with most of the previously mentioned and surveyed literature, we restrict ourselves in the following to one-dimensional models, i.e., $x \in \Re$, and assume a scalar control, $u \in \Re$. However, we will address extensions to the case $x \in \Re^2$. To simplify the notation, we omit arguments whenever possible without risk of confusion. In particular, the variables are not indexed with time t in the rest of the paper. Similarly, the word optimal will be typically omitted. Thus, for example, an optimal trajectory (optimal solution) will be termed trajectory (solution).

Much of the argumentation will be conducted in terms of the current value Hamiltonian $H(u, x, \lambda)$:

$$H(u, x, \lambda) \equiv f_0(u, x) + \lambda f(u, x). \tag{3}$$

In terms of this Hamiltonian, the first order conditions for an optimal policy u are:

$$\max_u H \overset{\text{for an interior solution}}{\Rightarrow} H_u = 0,$$

$$\dot{\lambda} = r\lambda - H_x \text{ and } \lim_{t\to\infty} \exp(-rt)\lambda = 0.$$

We assume throughout this article that $H_{uu} \leq 0$. We speak of a *concave* model if H is jointly concave in the state x and the control u, i.e., if $H_{uu}H_{xx} - H_{ux}^2 \geq 0$. Otherwise, we speak of a *non-concave* model if $H_{uu}H_{xx} - H_{ux}^2 \leq 0$, or of a *convex* model if $H_{xx} > 0$. Note, however, that we do not require concavity or convexity globally, but only over some compact set of interest for the concrete problem studied. Furthermore we assume, unless otherwise specified, that $H_{uu} < 0$. This insures that the equation $H_u = 0$ can be solved for the control.

Besides the Hamiltonian approach, another method based on the Hamilton-Jacobi-Bellman (HJB) equation is used at places, in particular, in Section 5. This approach is based on the fact that the value function V must satisfy the HJB functional equation:

$$rV(x) = \max_{u \in U} \left[f_0(x,u) + V'(x) f(x,u) \right]. \tag{4}$$

A third approach, discrete time dynamic programming, can be employed to study global dynamics and to detect thresholds based on a discretization of (4). Further details on these last two methods are given in Section 5.

4 Thresholds and necessary conditions

In this section, we classify and compare the necessary conditions for different types of thresholds separating optimal trajectories towards different steady-states. For the considered class of problems $P(\mathbf{a})$, convexity or at least non-concavity is usually considered to be the very property that causes the long-term behavior to depend upon the initial state, i.e., that leads to history dependence. This history dependence due to 'increasing returns', 'positive feedbacks', etc., plays a central role in many policy related discussions, ranging from the choice of a technology to differences in economic development, among others. The first subsection reviews this traditional road to multiple steady-states. The next subsection draws attention to the fact that this is not the only route and that a strictly concave framework does not rule out history dependent outcomes – a result, that may appear surprising and, in any event, has been largely overlooked if not negated in the literature. A separate subsection is devoted to the comparison of these two different backgrounds for multiple steady-states – the common denominator between the two being that the existence of an unstable steady-state is a necessary condition for history dependence. However, before comparing the two mechanisms, a subsection extends the analysis to the case of linear control models. The last subsection cursorily addresses extensions to higher dimensional systems.

4.1 Convexity and non-concavity

The theoretical contributions of Skiba (1978) and Dechert-Nishimura (1983) have sharpened our understanding of history dependent evolutions due to convexities and non-concavities. Our presentation of their results is based on the well known Ramsey model. In this model, u is consumption, f_0 the utility from consumption, x the capital stock, and δ is the capital depreciation rate. The problem $P(\mathbf{a})$ takes the form:

$$\max_u \int_0^\infty e^{-rt} \left(f_0(u)\right) dt, \tag{5}$$

$$s.t. \; \dot{x} = f\left(x, u\right) = F(x) - u - \delta x, \;\; x(0) = \mathbf{a}. \tag{6}$$

where F is the production function. The non-concavity arises through a local convexity of the production function, $F'' > 0$ for $x < \bar{x}$, $F'' < 0$ for $x > \bar{x}$, some $\bar{x} > 0$. This local convexity reflects increasing returns to scale for small capital stocks, $x < \bar{x}$.

The first order conditions yield the famous Ramsey rule:

$$F' = r + \delta \tag{7}$$

Since F' is monotonically declining and ranges over all positive real numbers for F satisfying the Inada conditions, the Ramsey rule defines a unique steady-state x^R in the standard case of global diminishing returns. However, if F is locally convex, it allows for two steady-states x_R and x^R, with $x_R < x^R$. The lower steady-state x_R lies in the convex domain of F, $x_R < \bar{x}$. The upper steady-state x^R lies in the concave domain, $x^R > \bar{x}$. Of these two steady-states, the one in the convex domain, x_R, is unstable. This gives rise to a threshold x^s that separates two domains of attraction: $x(t) \to x^R$ for $x_0 > x^s$, and (asymptotically for f_0 satisfying the Inada conditions) $x(t) \to 0$ for $x_0 < x^s$. Typically, the threshold lies in a vicinity of the unstable steady-state x_R.

Locally increasing returns to scale underlie many economic models and are arguably the most typical cause for convexity or non-concavity. The economic sources of these increasing returns have already been mentioned in section 2. They can, for example, arise from fixed costs in public infrastructure and networks such as telephone networks. They can give rise to multiple steady-states if 'average' costs are high for low values of the stock, that is e.g. if the telephone network deserves only a small number of users.

4.2 Thresholds in concave models

The existing literature strongly suggests that convexities (or at least of non-concavities) are necessary in order to obtain multiple steady-states. For example, Arthur (1989) states that increasing returns give rise to lock-ins and thus to history dependence, but that the outcome is independent of history if technologies are subject to constant or diminishing returns. The purpose of this subsection is to correct this overall perception by showing that multiple steady-states and history dependence are possible in strictly concave inter-temporal optimization problems. Very early examples for this can be found in Kurz (1968) and Liviatan and Samuelson (1969). However, they appear to have been neglected in most of the subsequent literature.

The following exposition draws on Wirl and Feichtinger (1999), that derives the following necessary condition:

$$\frac{d\dot{x}}{dx} = f_x + f_u u_x > 0 \text{ at a steady-state} \tag{8}$$

for the existence of an unstable steady-state and of a threshold within a concave framework. This condition can only be satisfied if at a steady-state either:

$$r > f_x > 0 \tag{9}$$

or:

$$f_u u_x = f_u \frac{-H_{ux}}{H_{uu}} > 0. \tag{10}$$

Condition (9) requires 'growth', i.e., $f_x > 0$, but below the rate of discount. Thus, the standard Ramsey and the standard renewable resource models do not allow for an unstable steady-state. Condition (10) also requires growth, but this growth is now indirectly induced by the optimal control. This requires that the mixed derivative characterizing the state-control interactions be the proper magnitude and sign: $H_{ux} > 0$ for $f_u > 0$, otherwise $H_{ux} > 0$. In this regard, note that Kurz (1968) emphasizes the 'growth' condition (9). Specifically, he introduces a wealth externality into the standard Ramsey model of optimal growth, that leads to an unstable steady-state with $r > f_x > 0$. On the other hand, while starting like Kurz with the Ramsey framework, Liviatan and Samuelson (1969) argue that an externality is not needed and rely on the control-state interactions (10) to insure the existence of an unstable steady-state.

To demonstrate the usefulness and the simplicity of the Wirl-Feichtinger approach, and in particular of the growth condition (9), consider the traditional Ramsey model with a strictly concave production function F. However, assume that the utility function f_0 now includes wealth effects as in Kurz (1968) and Wirl (1994):

$$\max_u \int_0^\infty e^{-rt} \left(v(u) + w(x) \right) dt, \tag{11}$$

$$s.t. \ \dot{x} = F(x) - u - \delta x, \ x(0) = \mathbf{a}. \tag{12}$$

In this formulation, $f_0 = v(u) + w(x)$. That is, total utility is the sum of utility from consumption $v = v(u)$, and of a wealth effect $w = w(x)$. The separable specification of f_0 and f rules out (10) as a source for an unstable steady-state. Without the wealth effect, the Ramsey rule implies that (9) would also not be satisfied. Indeed, the Ramsey rule requires $F' = r + \delta$, while (9) demands $F' < r + \delta$. The wealth effect, however, increases the stationary capital stock, thus decreasing F'. Thus, the model with wealth effect satisfies the growth condition $r > \frac{\partial \dot{x}}{\partial x} = F' - \delta > 0$ for any steady-state between the traditional Ramsey rule, $F' = r + \delta$, and the maximum sustainable consumption, $F' = \delta$. An steady-state in this range is a candidate for an unstable steady-state even if the model is concave.

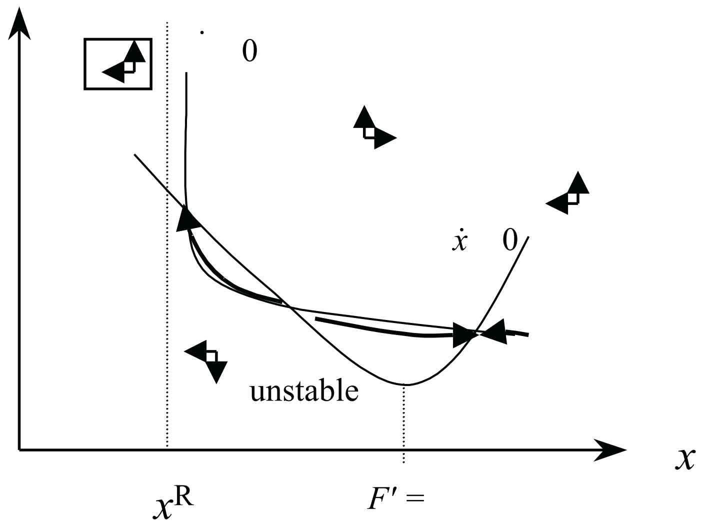

Figure 1: Phase diagram of the Ramsey model with wealth effects.

From the Hamiltonian:

$$H = v(u) + w(x) + \lambda \left[F(x) - u - \delta x \right] \tag{13}$$

one derives the first order conditions for interior solutions:

$$v' - \lambda = 0, \tag{14}$$

$$\dot{\lambda} = (r + \delta - F')\lambda - w'. \tag{15}$$

Substituting the optimal control determined by the maximum principle (14), $u = C(\lambda)$, $C' = \frac{1}{v''} < 0$, into the state differential equation yields the canonical equations in (x, λ) sketched in the phase diagram of Figure 1. The downwards sloping curve $\lambda = \frac{v'}{r+\delta-F'}$ characterizes the $\{\dot{\lambda} = 0\}$ isocline. The $\{\dot{x} = 0\}$ isocline is U-shaped with its minimum at the point where stationary consumption is maximized ($F' = \delta$), and poles at the points of zero consumption ($x = 0$ and $F(x) = \delta x$) for v satisfying the Inada conditions. This highlights that multiple steady-states may exist. In fact very simple numerical examples can be used to verify this existence – see e.g. Hof and Wirl (2000).

Some commentators have critically noted that conditions (8), (9) and (10) are only necessary. However, so are the familiar conditions of either convexity (with respect to the state) or lack of (joint) concavity. For example, the model of saving and growth with habit formation of Caroll, Overland and Weil (2000) has a globally non-concave utility function (a fact that, incidentally, is not mentioned by the authors). Yet, the long run outcome is unique.

4.3 Models linear in the control

A model that is linear in the control, $H_{uu} = 0$, has the same canonical equation system than in the previous case, with the only modification that the optimal control now depends discontinuously on the state and the co-states:

$$u = u(x, \lambda) = \begin{matrix} \bar{u} \\ \text{singular arc} \\ \underline{u} \end{matrix} \quad \text{if} \quad H_u = f_{0_u} + \lambda f_u \begin{matrix} > \\ = \\ < \end{matrix} 0,$$

where $\underline{u}$ and $\bar{u}$ denote lower and upper bounds on the control, $u \in U = [\underline{u}, \bar{u}]$. A steady-state is determined by the intersection of the singular arc, $H_u = 0$, with the $\dot{\lambda} = 0$ isocline. The singular arc is defined by $\lambda^{\text{sing}} = -f_{0_u}/f_u$. Because of $H_{uu} = 0$, a concave Hamiltonian implies $H_{ux} = 0$. Thus, λ^{sing} is a constant and the $\dot{\lambda} = 0$ isocline is monotonically declining. Consequently, multiple steady-states are impossible in a concave model, see Wirl and Feichtinger (1999).

Yet, if there is a local convexity with respect to the state, multiple steady-states are possible even in a separable model. Brock (1983) provides a nice example in the context of lobbying and entry deterrence. Moreover, even the milder condition of a lack of joint concavity allows for multiple steady-states and history dependence. Although concavity with respect to the state implies that the $\dot{\lambda} = 0$ isocline is monotonic, a singular arc depending on x may be sufficient for the isoclines to intersect more than once. This can be the case even if this dependence is linear, thus preserving concavity in x but not joint concavity in x and u. This can be shown on the following simple example of renewable resource extraction with an interaction in utility between the catch u and the biomass x:

$$f_0 = \alpha u + \beta x + \gamma u x \text{ and } f = g(x) - u, \; g(x) = x(1 - x), \qquad (16)$$

where $g(x)$ is the growth function for the biomass. The singular arc specifies the costate as a linear function of the state, $\lambda^{\text{sing}} = \alpha + \gamma x$. The differential equation for the costate is given by:

$$\dot{\lambda} = (r - g')\lambda - \gamma u - \beta.$$

Substituting the control that ensures a steady-state, $u = x(1-x)$, into the differential equation for the costate and solving for λ leads to:

$$\dot{\lambda} \mid_{\dot{x}=0} = 0 \Longleftrightarrow \lambda = \frac{\beta + \alpha g(x)}{r - g'}.$$

A steady-state is determined by the intersection of $\dot{\lambda} \mid_{\dot{x}=0} = 0$ with λ^{sing}. For a proper choice of the parameters, there can be two positive steady-states, see Figure 2.

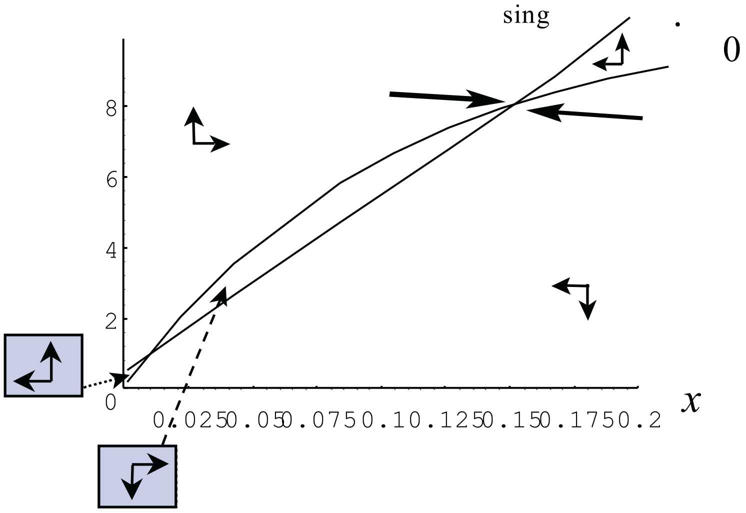

Figure 2: Phase diagram for (16) and the parameter values $r = 1.5$**,** $\alpha = 0.5$**,** $\beta = 0.05$**,** $\gamma = 50$**.**

4.4 Common features and differences

The economic implications of multiple equilibria, as we just discussed them, appear largely independent of the question whether or not the underlying model is concave. In any case, if there are multiple steady-states, there exists a threshold (in the form of a Skiba point or a Skiba curve), that is, a (set of) critical values of the state x with the following property: The optimal policy is different depending on which side of the threshold the current state lies. In the case of a one-dimensional system, the optimal policy will thus be to let x grow if its current value lies on the one side, to let it decline if it lies on the other side of the Skiba point. This may possibly lead to an optimal steady-state that is "inefficient" (i.e., if x is a positively valued stock) or "boundary" (e.g. $x = 0$).

However, there are important differences between the unstable steady-states that arise in the one or the other case. In order to recognize them, let us first investigate the eigenvalues of the Jacobian of the canonical equations. The eigenvalues of this Jacobian are always real for a concave Hamiltonian, see Wirl and Feichtinger (1999). Thus, in the concave case, any unstable steady-state is a node. By contrast, much of the literature on non-concave applications describes the unstable steady-states as spirals. Thus, one might be led to believe that the type of local dynamics around the unstable steady-states, node or spiral, allows for properly differentiating between the concave and the non concave/convex cases. Yet this is not the case. Non-concavity or convexity allows for complex eigenvalues and thus for a spiral, but does not rule out a node.

Another feature emphasized throughout the literature is that history dependence involves a jump in the control at the threshold. Yet, a jump is impossible within a concave framework, since the policy must be unique. Hence, the existence a jump seems to be a truly distinctive characteristic of non-concave and convex models, as opposed to concave ones.

This point can be best illustrated in terms of the value function, $V(\mathbf{a})$. Suppose there are three steady-states, $x_1 < x_2 < x_3$, with x_2 unstable. In the case of a concave framework, the value function is unique, see the left drawing in Figure 3. Therefore, in particular, there is an unique optimal control associated with the unstable steady-state x_2. Moreover, x_2 is optimal. By contrast, in a non-concave or convex framework, at least two value functions, say V_1 and V_3, exist, the first being associated with x_1 and the second with x_3. Since the problem of interest is a maximization, one should choose for any given initial state $\mathbf{a}$ the solution that yields the highest possible payoff. If $V_1(\mathbf{a}) < V_3(\mathbf{a})$, then it is optimal to choose the optimal control path that leads ultimately to x_3, and if $V_1(\mathbf{a}) > V_3(\mathbf{a})$, the one leading to x_1. At the value $\widehat{\mathbf{a}}$ of the initial state for which V_1 and V_3 intersect, i.e. for which $V_1(\widehat{\mathbf{a}}) = V_3(\widehat{\mathbf{a}})$, one is indifferent between heading towards x_1 or towards x_3. If indeed several value functions do exist, the threshold value $\widehat{\mathbf{a}}$ will only incidentally coincide with the unstable steady-state x_2. Moreover, x_2 will not be optimal. This can be seen by substituting the stationary control satisfying the first-order conditions associated with x_2 into the steady-state equation. The payoff at this steady-state falls short of the maximum, see the right drawing in Figure 3. Since the value functions cross at $\widehat{\mathbf{a}}$, the derivatives of the value functions will typically differ. But, according to the Hamilton-Jacobi-Bellman equation, the optimal control depends on this derivative. Thus, it jumps at $\widehat{\mathbf{a}}$.

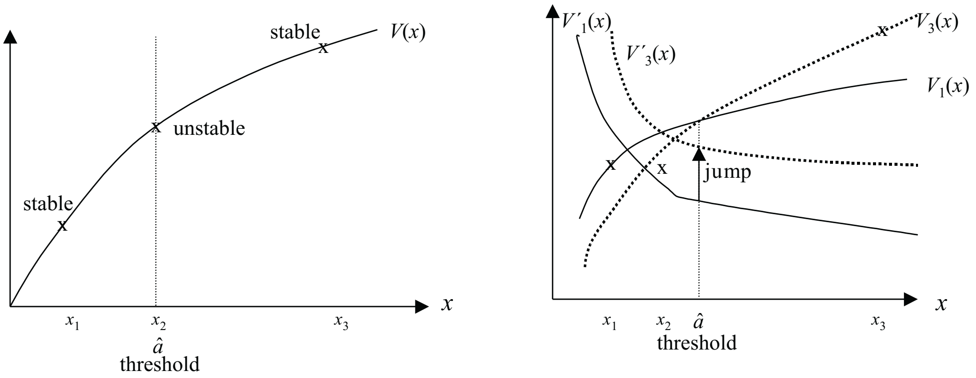

Figure 3: Camparing unique (concave) and multiple value functions

Concavity insures that there is an unique optimal policy. By contrast, non-concavity or convexity always imply multiple policy functions when the unstable steady-state is a spiral: a simple inspection of the phase diagram shows that the same policy function cannot continuously connect the steady-states. However, matters are less clear when the unstable steady-state is a node. For example, consider the relative adjustment cost framework in Hartl, Kort, Feichtinger and Wirl (2000):

$$\max_u \int_0^\infty e^{-rt}[v(x) - C(u/x)]dt,$$

$$\dot{x} = u - \delta x, \;\; x(0) = \mathbf{a},$$

where v is the concave gross profit function, x the capital stock, u the gross investment, C a convex costs function with the ratio of replaced capital as argument. This framework seems particularly well suited to trace out the rather subtle points we are addressing here, because (a) it insures the existence of multiple equilibria (or example, for the quadratic specification $v = x - \frac{1}{2}x^2$ and $C = \frac{1}{2}\gamma\left(\frac{u}{x}\right)^2$, the model admits three steady-states) and (b) the unstable equilibrium can fall into the concave or the non-concave domain and be either a node or a spiral. In the case of an unstable node in the non-concave domain, Hartl, Kort, Feichtinger and Wirl (2000) present a numerical example with a phase diagram that allows for a continuous connection between the steady-states, see Figure 4. Thus, for specific values of the parameters, a unique value function may indeed exist. More precisely, while the uniqueness of V is guaranteed by concavity, but often violated in the convex case, it can not be excluded in the presence of non-concavities or convexities, presumably at least as long as these remain mild. This point clearly requires further research. In any event, it suggest the need for correcting the loose and often incorrect statements found in the literature where, typically, an unstable spiral is assumed without carrying out the necessary eigenvalues analysis.

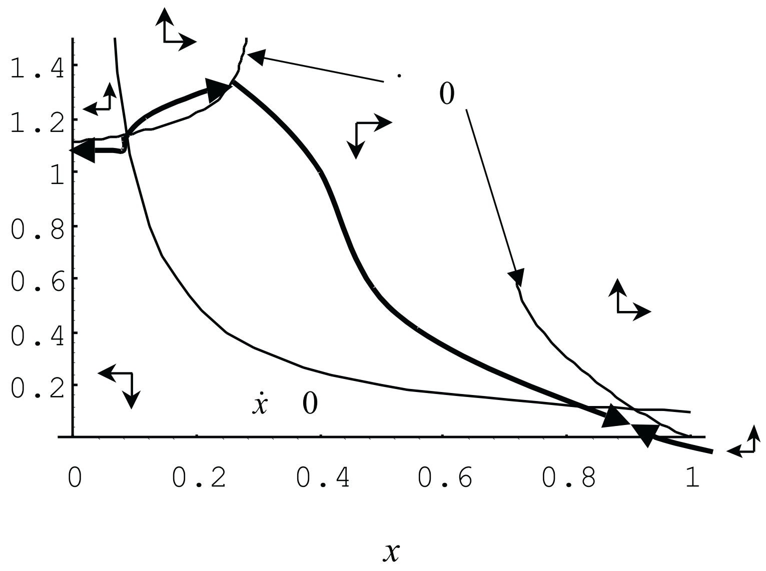

Figure 4: Phase Diagram for the relative adjustment cost framework, quadratic specification, $\gamma = \frac{3}{2}$, $r = 1$, $\delta = 0.1$.

For the concave variant of the above model the value function and the threshold – which in this case coincides with the middle unstable equilibrium – are computed using the HJB-equation; see appendix 1.

4.5 Higher order systems

We consider only such two-dimensional systems, $x \in \Re^2$, that can be derived from one-dimensional ones using the embedding approach developed in Feichtinger, Novak and Wirl (1994). With this approach, the originally one-dimensional problem is transformed into a two-dimensional one by introducing control adjustments costs, denoted v. For simplicity's sake, these costs are assumed here to be quadratic:

$$\underset{u(t)\in U}{Max} \int_0^\infty e^{-rt}\left(f_0(x,u) - \frac{1}{2}cv^2\right)dt, \tag{17}$$

$$s.t.\ \dot{x} = f(x,u),\ x(0) = \mathbf{a},\ \dot{u} = v,\ u(0) = u_0, \tag{18}$$

Since $v = 0$ at a steady-state, the original one-dimensional and the derived two-dimensional problems both have the same steady-states. Moreover, an unstable steady-state of the original problem remains unstable for the derived problem, since $\det(J) > 0 \Longleftrightarrow \det(\widetilde{J}) < 0$, where J is the determinant of the canonical equations system of the original, and $\widetilde{J}$ the Jacobian of the four-dimensional canonical equations system of the derived problem. It is impossible to stabilize an unstable system by introducing adjustment costs.

Interestingly enough, the converse is not true. Under certain conditions, adjustment costs may destabilize an otherwise stable steady-state, see Feichtinger, Nowak, and Wirl (1994). In particular, in the concave case, adjustment costs can transform a formerly stable steady-state into an unstable steady-state or a limit cycle if the growth condition $r > f_x > 0$ is satisfied. This may appear counter-intuitive, since at a stable steady-state no control changes are necessary, while the destabilization (that may lead e.g. to a Hopf cycle) requires the permanent use of costly control changes.

Thus, the derived problem can have an unstable steady-state either because the original problem had one, or because the adjustment costs destabilize an originally stable steady-state. The growth condition is necessary in both cases. However, the second road to instability does not lead to thresholds and history dependence. These phenomena arise only in the case where the original steady-state is unstable, $\det(J) > 0$, which as previously indicated is the case if and only if $\det(\widetilde{J}) < 0$.

The condition for an unstable steady-state, $\det(\widetilde{J}) < 0$, is equivalent to three eigenvalues of the Jacobian being either positive or having positive real parts, with the fourth one being negative, see Dockner (1985). Hence, an unstable steady-state remains conditionally stable along a one-dimensional manifold M of initial conditions. This is illustrated in Figure 5, that shows a situation with two (saddle-point) stable steady-states (x) and an unstable one. The unstable steady-state can be reached from any initial condition along the dotted line, that is, from any point on the manifold M. This extends by one dimension the well-known property that in the one-dimensional case the system remains in the unstable equilibrium if it starts there. In a concave framework, the unstable steady-state is optimal, so that the manifold M is also the threshold that separates the domains of attractions of the stable steady-states. The points on this threshold are 'indeterminate' in the sense that, starting from any of these points, it does not matter towards which of the equilibria one converges. This indeterminacy is of measure zero and can therefore be ignored in a discussion of generic outcomes, which are necessarily defined in terms of stable steady-states.

For a convex model, similarly to the one-dimensional case, the threshold is given by the intersection of the two value functions associated with the long-run outcomes – that is, with the two points marked $\times$ in Figure 5. The projection of this intersection onto the state space, a one-dimensional manifold, is now the threshold that separates the domains of attractions of the stable steady-states. This threshold can cross the one-dimensional manifold M, if the unstable steady-state and the corresponding stationary control are not optimal.

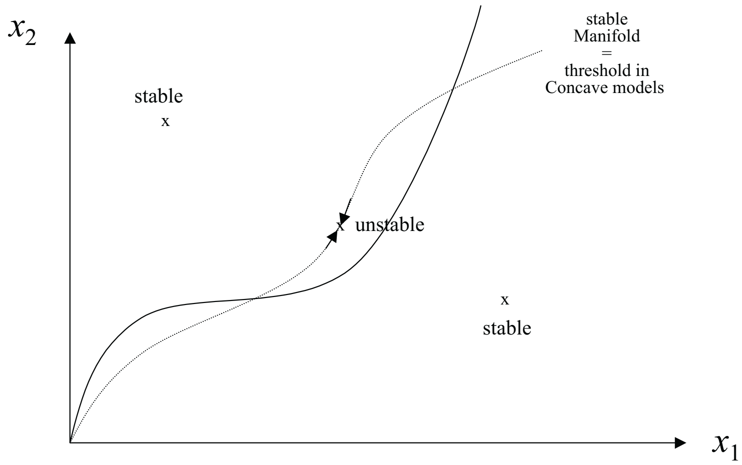

Figure 5: Phase Diagram for the relative adjustment cost framework, quadratic specification, $\gamma = \frac{3}{2}$, $r = 1$, $\delta = 0.1$.

Thus, the concept of a separating threshold between stable equilibria defined for one-dimensional models extends naturally, and directly to the case of two dimensions. The main difference is that it is no longer a point, but a curve, that separates the domains of attractions. Finding this separating curve in the concave case is fairly straightforward, but not trivial, since it is the stable manifold associated with the unstable equilibrium. If the model is not concave, its computation is even more involved, as shown in the next section.

5 Numerical Methods for Detecting Thresholds

A rigorous study of a dynamic model with multiple steady-states would require locating the Skiba points **a** analytically. Unfortunately, this appears to be impossible when the Skiba points do not coincide with unstable steady-states, due to the lack of an appropriate "local" equation to define them. Thus, the Skiba points have to be determined numerically by either one of the three methods presented in this section.

The **first method** uses the Hamilton-Jacobi-Bellman (HJB) equation to solve $P(\mathbf{a})$ numerically. We summarize the corresponding algorithm as applied by Semmler and Sieveking (1999).Two examples are computed in the appendix. The algorithm, that is due to Brooks Ferebee, in general implies three steps:

1. Compute the candidates for equilibrium steady states by solving for the stationary HJB-equation

$$f_0(e) = \max_{u \in U} \left[f_0(e,u) + \frac{1}{r} f_0'(e) f(e,u) \right], \tag{19}$$

where $f_0'(x)$ is the derivative of $f_0(x)$ with respect to x, and where e is a steady-state – and thus, a candidate for a long-run optimal steady-state (remember that every optimal steady-state e solves (19) , but not every steady-state e is optimal).

2. Solve the dynamic HJB-equation

$$rV(x) = \max_{u \in U} \left[f_0(x,u) + V'(x) f(x,u) \right] \tag{20}$$

by starting with the equilibrium candidates e as initial conditions. To obtain $V'(x)$ explicitly as a function of x and $V(x)$ compute

$$V'(x) = G(V(x), x), \tag{21}$$

The initial value problem

$$\begin{aligned} V'(x) &= G(V(x), x), \\ V(\mathbf{a}) &= \int_0^\infty e^{-rt} f_0(\mathbf{a}, u) dt = \frac{1}{r} f_0(\mathbf{a}, u), \end{aligned}$$

to be solved provides the solutions of (21). Then index with i the solutions obtained for different candidates e.

3. For every x compute V_i and then set:

$$V(x) = \underset{i}{Max} V_i \tag{22}$$

V is the desired value function indicating thresholds where the piece-wise obtained value functions intersect.

All three steps are illustrated by Figure 6, which shows the piece-wise solution of the value function as indicated in step 3. The Skiba point x^s is located at the intersection of the value functions V_1 and V_2. Note that, as it is the case in Figure 6, the Skiba point does not necessarily coincide with a candidate e.

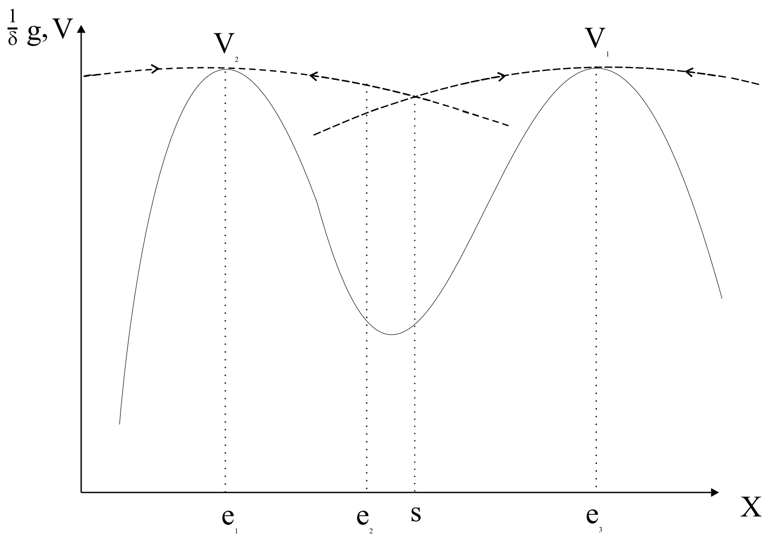

Figure 6: Skiba point and global dynamics computed through the HJB-equation.

The main achievement of this algorithm is to find the location of the Skiba points from the solutions of the HJB-equation. The outer envelop defined by (22) determinates the optimal solutions and thus the optimal global dynamics, that is, the history-dependent solutions. Note that knowing V permits to calculate the optimal control $u(x)$ in feedback form using the HJB-equation. Finally, remark that the policy function might not be continuous at the Skiba-point, see section 4.4. and Figure 3.

The **second method** that can be employed is based on the maximum principle and the Hamiltonian. Usually, the Hamiltonian does not allow to recognize the globally optimal steady-states, since it generates only necessary conditions. In the case of three candidates, these necessary conditions typically imply the dynamic properties shown in Figure 7.

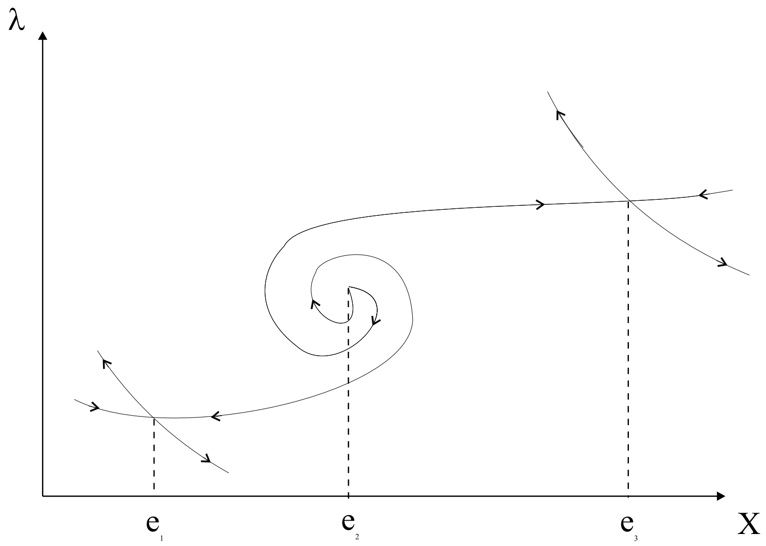

Figure 7: Local dynamics about the candidates

Usually the equilibrium candidate e_1 is a saddle-point, e_2 is unstable node or focus, and e_3 is again a saddle-point. There are connecting orbits from the candidate e_2 to the other candidates e_1 and e_3. Thus, in general, one can proceed as follows to obtain the global dynamics from the Hamiltonian $H(\cdot)$ associated with the problem $P(\mathbf{a})$.

1. Compute the candidates e from the optimal control (or co-state) equations and state equations.

2. Compute the local dynamics about the candidates.

3. Compute the integrals along the stable manifold from the right and from the left of the middle unstable steady-state. The intersection of the two integral curves is the Skiba point, as shown in Figure 8.

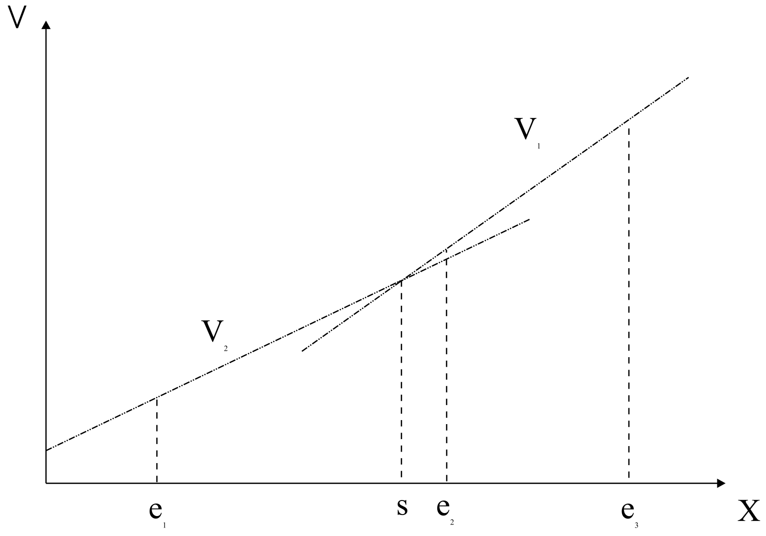

Figure 8: Skiba points and global dynamics computed through the Hamiltonian Local dynamics about the candidates

The point where the two integral curves intersect is the Skiba point because at this point the pay-offs for going to e_1 are the same as for going to e_3. At the middle candidate e_3 there are typically two solutions satisfying the first-order conditions for optimality. However, one is superior to the other. For details, see Beyn, Pampel and Semmler (2000), that also indicate how to compute thresholds in higher dimensional systems.

This second method has been suggested by Skiba (1978), analytically further pursued by Brock and Malliaris (1996) and Brock and Starret (1999), and numerically implemented by Beyn, Pample, and Semmler (2000), and by Haunschmied, Kort, Hartl, and Feichtinger (2000). Although it is useful for computing the global dynamics, it has shortcomings. The precision with which the Skiba point can be computed depends crucially on the approximation of the connecting orbits, that is, of the stable manifolds for the candidates e_1 and e_3 as shown in Figure 7. In order to obtain the correct integrals, the connecting orbits should be precisely computed on grid points in the state space, starting from e_2 and moving to the left to e_1 and to the right of e_2 to e_3.

A **third method,** dynamic programming, can also be employed to compute the Skiba points. Using the continuous dynamic programming equation (20) is equivalent to iterating on the value function. If the iteration is properly done and converges, the value function will be greater at a non-optimal steady-state than the value obtained from the stationary control at the candidate. That is, the optimality of the candidate can be checked by direct inspection.

Generically, however, the dynamic programming method is more efficient at finding strong attractors (for example limit cycles) than thresholds or Skiba points, see Sieveking and Semmler (1997). Strong attractors are not influenced severely by numerical errors such as rounding inaccuracies. By contrast, the search for a Skiba point can be considerably affected, as this search amounts to numerically locate in a one-dimensional state space a point where the control u starts changing direction or jumps as in the case of a discontinuous control. Yet, the use of dynamic programming on a grid for the state and control equations generates numerical rounding errors that pile up in the iteration of the value function, and also impact the control u. To rely on dynamic programming to numerically find the Skiba points, it is necessary to have trustworthy estimates of the associated error bound and a very refined algorithm. The problems of discretization and estimation of error bounds are discussed in Semmler and Sieveking (1999a), who also give an example of the difficulty to find a threshold through dynamic programming. In any event, it is necessary to point out that results obtained from dynamic programming may be less reliable than the results obtained from first the two methods.

6 Conclusions

Multiple equilibria constitute a low level, manageable form of complexity. A decision-maker, or a researcher, may have difficulty grasping and coping with, say, the full richness of chaotic non-linear dynamics. Multiple equilibria, that confront him with a finite set of well defined alternatives, offer a much simpler basis for reflection and action. At the same time, however, they seriously undermine the historical determinism that underlies many of the standard models of economics.

This paper has given a state-of-the art review of the conditions under which multiple equilibria can arise in representative agents perfect foresight dynamic optimization models. It has developed a typology of these conditions, clarified the properties of equilibria associated with different necessary conditions,and presented numerical approaches to study the models' global dynamics and Skiba thresholds. One of its main contributions, furthermore, is to have stressed a largely ignored fact. Even if there are no externalities, perfect foresight, and strict convexity (the economists' workhorse in insuring uniqueness of optimal solutions), multiple equilibria are possible. This suggests that history dependence may be a much more pervasive phenomenon in economics than usually assumed. Even in a very well behaved world, the far future may be very different depending on the current conditions. Since the latter are always subject to accidental events, otherwise similar economies need not systematically take the same road.

The results presented here are valid for centrally planned or representative agents economies. If there are heterogenous agents, the relationship between individual optimal and aggregate behavior can be different, and possibly more complicated, than described here. While all evidence shows that taking into account agents' heterogeneity increases, if anything, the scope for multiple steady-states, their proper modeling and analysis is the subject matter of future research.

A Computing the Value Function by Using the HJB-Equation

We want to demonstrate the usefulness of the HJB-equation to compute the value function and the threshold (Skiba point) that separate different domains of attraction. We present two examples. Both examples represent models of optimal investment where relative adjustment cost gives rise to multiple steady state equilibria. The second model, however, permits borrowing from capital market and considers the budget constraint of the firm.

A.1 Optimal Investment with Adjustment Cost

Our first example builds on the model of section 4.4. Details of this model can be found in Feichtinger et al (2000) and the computation of the value function and the thresholds for this model is undertaken in Kato and Semmler (2001).

The present value problem to be solved is as follows

$$V(x) = \max_u \int_0^\infty e^{-rt}[v(x) - C(u/x)]dt, \tag{A.1}$$

$$\dot{x} = u - \delta x, \ x(0) = \mathbf{a}, \tag{A.2}$$

where v is a concave gross profit function, x a stock variable , u the increase of the stock, $C(u/x)$, a convex cost function with relative adjustment cost as argument. Feichtinger et al (2000) take quadratic specifications $v = x - \frac{1}{2}x^2$ and $C = \frac{1}{2}\gamma\left(\frac{u}{x}\right)^2$. The model admits three steady-states and the unstable equilibrium can fall into the concave domain.

The HJB-equation of section 5 has, for the present model, the form

$$rV(x) = \max_u \left[(v(x) - C(u/x)) + V'(x)(u - \delta x)\right] \tag{A.3}$$

We can compute the value function and thresholds for the solution of this problem in three steps.

Step 1: Compute the steady states for the stationary HJB-equation

If e is an equilibrium then

$$u - \delta x = 0$$

and

$$rV(e) = v(e) - C(\delta). \tag{A.4}$$

Thus

$$V(e) = \frac{1}{r}\left[v(e) - C(\delta)\right]$$

and

$$V^{'}(e) = \frac{1}{r}v^{'}(e). \tag{A.5}$$

The equilibrium e satisfies

$$rV(e) = \max_{u}\big[v(e) - C\big(\frac{u}{e}\big) + V^{'}(e)(u - \delta e)\big]. \tag{A.6}$$

Substituting (A4) and (A5) into (A3) yields

$$v(e) - C(\delta) = \max_{u}\big[v(e) - C\big(\frac{u}{e}\big) + \frac{1}{r}v^{'}(e)(u - \delta e)\big].$$

Solving $\frac{d}{du}\big[v(e) - C\big(\frac{u}{e}\big) + \frac{1}{r}v^{'}(e)(u - \delta e)\big] = 0$ gives

$$-rC^{'}\big(\frac{u}{e}\big)\frac{1}{e} + v^{'}(e) = 0. \tag{A.7}$$

From the specific functions for (A1), (A2) we obtain

$$v'(x) = 1 - x \tag{A.8}$$

$$C^{'}\big(\frac{u}{x}\big) = \gamma\big(\frac{u}{x}\big). \tag{A.9}$$

The equilibrium condition (A7) becomes

$$-r\gamma\big(\frac{u}{e}\big)\frac{1}{e} + 1 - e = 0 \tag{A.10}$$

or

$$u = \frac{1-e}{rc}e^{2}. \tag{A.11}$$

Inserting this condition into the steady state condition we obtain three steady state equilibria from

$$\begin{aligned} \dot{x} = u - \delta e \;&=\; \frac{1-e}{r\gamma}e^{2} - \delta e \\ &=\; e\big[\frac{1-e}{r\gamma}e - \delta\big]. \\ &=\; 0 \end{aligned} \tag{A.12}$$

since this implies
$e = 0$ or

$\left[\frac{1-e}{r\gamma}e - \delta\right] = 0$, that is $e^2 - e + r\gamma\delta = 0$.
Thus, the optimal three steady states are

$$e = \begin{cases} 0 \\ \frac{1\pm\sqrt{1-4r\gamma\delta}}{2} = \frac{1\pm\sqrt{D}}{2} \end{cases} \tag{A.13}$$

where we assume $D \equiv 1 - 4r\gamma\delta \geq 0$.
For steady states both of the following two conditions are satisfied:

$$u = \frac{1-e}{r\gamma}e^2 \tag{A.14}$$

$$u = \delta e \tag{A.15}$$

For $\gamma = 1.5$, $\delta = 0.1$ and $r = 1$ we obtain the following three solutions as candidates for optimal equilibria: $x^* = 0$, $x^{**} = 0.184$, $x^{***} = 0.816$.

Step 2: Solve the dynamic HJB equation starting from the equilibrium candidates.
Using the stationary HJB equation again we obtain

$$\max_u = [e - \frac{1}{2}e^2 - \frac{1}{2}\gamma(\frac{u}{e})^2 + V'(e)(u - \delta e)]. \tag{A.16}$$

Solving $\frac{d}{du}[e - \frac{1}{2}e^2 - \frac{1}{2}\gamma(\frac{u}{e})^2 + V'(e)(u - \delta e)] = 0$ gives

$$-\gamma(\tfrac{u}{e})\tfrac{1}{e} + V'(e) = 0$$

or

$$\frac{V'(e)e^2}{\gamma} = u \tag{A.17}$$

Substituting (A17) into (A16) gives

$$rV(e) = e - \frac{1}{2}e^2 + \frac{1}{2}\frac{e^2}{\gamma}V'(e)^2 - \delta eV'(e), \tag{A.18}$$

therefore

$$V'(e)^2 - 2\frac{\gamma\delta}{e}V'(e) + 2\frac{\gamma}{e} - \gamma - 2\frac{\gamma r}{e^2}V(e) = 0. \tag{A.19}$$

Then we obtain an ordinary differential equation in V with candidates of steady states as initial condition.

$$V'(e) = \frac{\gamma\delta}{e} \pm \sqrt{(\frac{\gamma\delta}{e})^2 - (2\frac{\gamma}{e} - \gamma - 2\frac{\gamma r}{e^2}V(e))}. \tag{A.20}$$

Using this information for the solution of V we get

$$V'(x) = \frac{\gamma\delta}{x} - \sqrt{(\frac{\gamma\delta}{x})^2 - (2\frac{\gamma}{x} - \gamma - 2\frac{\gamma r}{x^2}V(x))} \quad \text{for} \quad x \leq e \tag{A.21}$$

$$V'(x) = \frac{\gamma\delta}{x} + \sqrt{(\frac{\gamma\delta}{x})^2 - (2\frac{\gamma}{x} - \gamma - 2\frac{\gamma r}{x^2}V(x))} \quad \text{for} \quad x < e \tag{A.22}$$

with

$$V(e) = \frac{1}{r}[e - \frac{1}{2}e^2 - \frac{1}{2}\gamma\delta^2]$$

as initial condition.

We solve the ODE in V by using the Euler method starting at e.

Step 3: Solve the global value function. We can compute the global value function for the original problem by

$$V(x) = \max V_i \tag{A.23}$$

The results of the piece-wise computation of the value function is shown in figure A1.

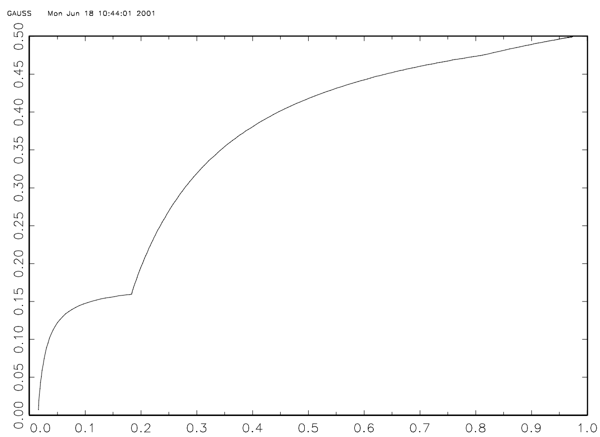

Figure A1:

In this model since the steady state $x^{**} = 0.184$ is a node, the thresholds coincide with the equilibria $x^{**} = 0.184$ and $x^{***} = 0.816$.

A.2 Optimal Investment, Adjustment Cost and Credit Market

The next example permits, beside adjustment cost, borrowing by the firm from capital markets and considers more explicitly the budget constraint of the firm; for details, see Semmler and Sieveking (2000). The present value problem is

$$V(k) = \underset{j}{Max} \int_0^\infty e^{-rt} f(k,j) dt \tag{A.24}$$

$$\text{s.t. } \dot{k} = j - \sigma k, \qquad k(0) = k_0 \tag{A.25}$$

$$\dot{B} = rB - f(k,j), \qquad B(0) = B_0 \tag{A.26}$$

where k, the capital stock, j, investment, B, debt of the firm and a net income function

$$f(k,j) = k^\alpha - j - j^2 k^{-\gamma} \tag{A.27}$$

with non-linear adjustment cost of capital. Here all variables are in efficiency units.

Moreover, $\sigma > 0, \alpha > 0, \gamma > 0$ are constants: The no-Ponzi game condition for this problem is

$$\lim_{t\to\infty} B(t) e^{-rt} = 0 \tag{A.28}$$

In order of the intertemporal budget constraint to hold it requires $B \leq V(k)$. This is needed for the problem to have a solution, see Semmler and Sieveking (2000).

The HJB-equation for this problem reads

$$rV = \max_j \left[k^\alpha - j - j^2 k^{-\gamma} + V'(k)(j - \sigma k)\right] \tag{A.29}$$

Here too, we can compute the value function and thresholds in three steps

Step 1: Compute the steady state candidates

Again note that for the steady state candidates, for which $0 = j - \sigma k$ holds, we obtain:

$$V(k) = \frac{f(k,j)}{r} \tag{A.30}$$

$$V'(k) = \frac{f'(k,j)}{r} = \frac{\frac{\partial}{\partial k}(k^{\alpha} - \sigma k - \sigma^2 k^{2-\gamma})}{r} \tag{A.31}$$

Using the information of (A30)-(A31) in (A29) gives, after taking the derivatives of (A29) with respect to j, the steady states for the stationary HJB equation:

$$-1 - 2jk^{-\gamma} + \frac{\alpha k^{\alpha-1} - \sigma - \sigma^2(2-\gamma)k^{1-\gamma}}{r} = 0 \tag{A.32}$$

Note that hereby $j = \sigma k$. The equation admits three steady states.

Step 2: Derive the differential equation V'

We derive the differential equation V$'$ by taking

$$\frac{\partial rV}{\partial j} = 0;$$

We obtain

$$-1 - 2jk^{-\gamma} + V'(k) = 0$$

Solving for the optimal j and using the optimal j in (A29) we get

$$V' = 1 + 2\sigma k^{1-\alpha} \pm \sqrt{(1 + 2\sigma k^{1-\alpha})^2 + 4\delta k^{-\alpha} V + k^{\gamma-\alpha} - 6} \tag{A.33}$$

Next, we start the iteration with steady states as initial conditions. For e, a steady state, we get as initial value for the solution of the differential equation (A33):

$$\begin{aligned} V_0 &= \int_0^{\infty} e^{-\delta t} g(e,j)dt \\ V_0 &= \frac{1}{\delta} g(e,j) \end{aligned}$$

Step 3: Compute the global value function by taking

$$V(k) = \max_i V_i$$

where $V(k)$ is the outer envelop of the piece-wise value function.

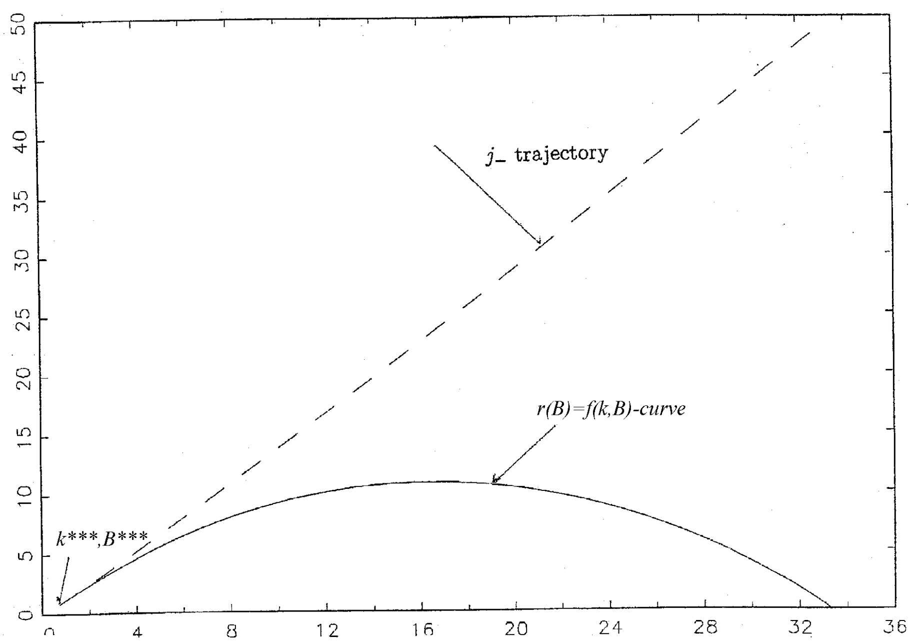

Figure A2a:

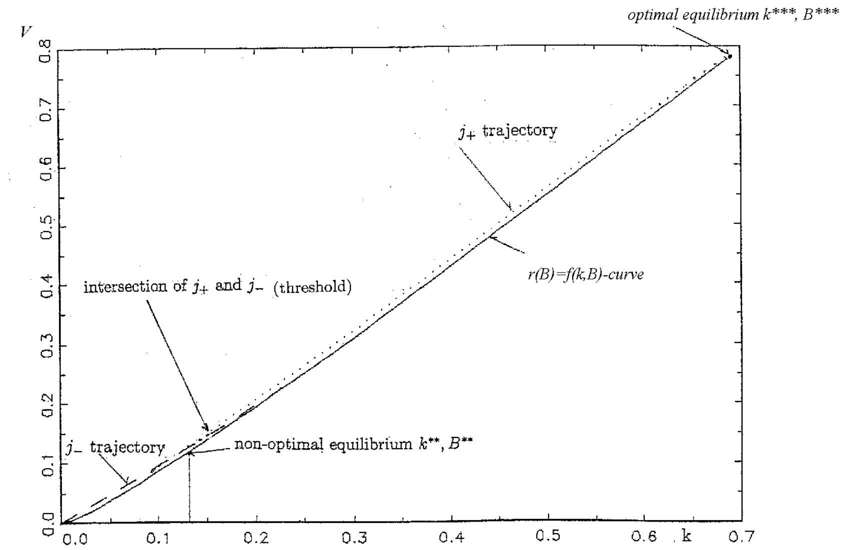

Figure A2b:

The results of the piece-wise computation of the value function are shown in figures A2a and A2b. A2b shows the trajectory to the right of the steady state, k^{***} and figure A2a the value function to the left of k^{***}. The two value functions to the left of k^{***} , the one starting at k^{***} and going to the left and the other starting at k^{*} going to the right, intersect. This intersection represents the Skiba-point.

Extension: If one allows for endogenous credit cost one has to compute piecewise the value function for

$$\begin{aligned}
\dot{k} &= j - \sigma k \\
\dot{B} &= H(k, B) - f(k, j) \\
f(k, j) &= k^{\alpha} - j - j^{\beta} k^{-\gamma}
\end{aligned}$$

with $H(k, B)$ the endogenous credit cost, depending on assets and liability of the firm (networth as collateral for the firm). This extension is presented in Semmler and Sieveking (2000). It leads to the following equation for candidates of equilibrium steady states for $H(k, B) = h(B) = rB^{\kappa}$ for $\kappa \geq 1$.

$$1 + 2jk^{-\gamma} = \frac{\alpha k^{\alpha-1} - \sigma - \sigma^2(2-\gamma)k^{1-\gamma}}{r\kappa(k^{\alpha} - \sigma k - \sigma^2 k^{2-\gamma})^{(\kappa-1)/\kappa}} \tag{A34}$$

Note that the steady state candidates are the same as in (A32) if in (A34) $\kappa = 1$. For details of the solution and the use of the HJB-equation to solve for thresholds, see Semmler and Sieveking (2000).

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