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An Evolutionary Portfolio Theory

by

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Abstract

The purpose of this paper is to suggest a new theory of portfolio selection which is based on evolutionary reasoning in simple repeated market situations. According to this new point of view the ultimate success of a portfolio strategy is measured by the wealth share the strategy is eventually able to conquer in an evolutionary process of market selection. We identify a simple portfolio strategy as being the unique evolutionary stable strategy.

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1 Introduction

The standard theory of portfolio selection (Markowitz 1952) provides an intuitive model of return and risk in terms of mean and variance. This model is very useful to guide one's intuition, and because of its appealing simplicity it is also commonly used in practical financial decisions. Moreover, ever since the work of Sharpe (1964), Lintner (1965) and Mossin (1966) the resulting consequences for capital market equilibrium have been explored and culminated in the celebrated CAPM.

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While the Markowitz model is often used in two period portfolio decisions its generalization to multiple periods, see e.g. Merton (1973), Breeden (1979), and Magill and Quinzii (2000), is still under debate. Alternatively it has been suggested to maximize the expected growth rate of wealth. In a series of papers, Hakansson (1970), Thorp (1971), Algoet and Cover (1988), and Karatzas and Shreve (1998), among others, have explored the maximum growth perspective. Computing the maximum growth portfolio is a non-trivial problem. Even if one restricts attention to simple trading strategies when markets are incomplete there is no explicit solution to this problem. By now numerical algorithms to compute the maximum growth portfolio have been provided by Algoet and Cover (1988) and Cover (1984, 1991), but so far practical decisions are rarely based on these ideas. Moreover, as usual in mathematical finance, the price process is taken to be exogenously given and consequences on the equilibrium are ignored following this approach.

We intend to contribute to this challenging problem by applying recent ideas from evolutionary game theory. The evolutionary approach to portfolio selection also takes a long-run perspective, however, it emphasizes market interaction. According to this approach portfolio strategies compete for market capital—the endogenous price process is thus a *market selection mechanism* along which some strategies gain market capital while others lose. The equilibrium notion this approach provides is a distribution of market capital (wealth shares) that is invariant under the market selection process. The species among which the endogenous price process selects are *portfolio rules* which have also been called *simple trading strategies.* These are non-negative vectors of expenditure shares for assets which are held constant over time. It is easy to see that every *monomorphic* population, i.e. a collection of traders using the same portfolio rule, gives rise to an invariant distribution of wealth shares. Hence the notion of an invariant distribution does not a priori select any particular portfolio rule. However, if one also checks the robustness of the invariant distributions with respect to the innovation of new portfolio rules (called mutations), then one can identify a single portfolio rule that is the unique *evolutionary stable strategy*, i.e. that is able to drive out any mutations.

According to this rule one should divide wealth proportionally to the expected relative returns of the assets. In the case of diagonal securities[1] the unique evolutionary stable strategy boils down to a well known trading strategy that in this case is known to be the best trading strategy in models with exogenous prices as well as in those with with endogenous prices: With

[1]We call a system of securities *diagonal* when in each state exactly one asset has a non-zero payoff. An example for these are Arrow-securities.

exogenous prices (fixed to one for every asset) it has been shown by Breiman (1961) that dividing income proportionally to the probability of the states maximizes the growth rate of wealth. This rule is known as "betting your beliefs." It can be generated by maximizing the expected logarithm of relative returns which in turn is known as the *Kelly rule*, Kelly (1956). With endogenous prices and diagonal securities, Blume and Easley (1992) have shown that "betting your beliefs" is the global attractor of the market selection mechanism. That is to say, this rule will have the highest growth rate of wealth in any population of portfolio rules. Blume and Easley (1992)'s result lays the foundation for an evolutionary portfolio theory because it is the first result that takes market selection seriously.

Our paper on the one hand extends Blume and Easley's model to any complete or incomplete payoff structure. On the other hand our result relies on the idea of evolutionary stability which so far has not been used in any portfolio theory. As in Blume and Easley (1992) we consider an economy with short lived assets. With regards to possible applications of the theory, this is an unsatisfactory assumption that will have to be generalized in future research. Moreover, we have borrowed the idea of simple trading strategies from Blume and Easley (1992), and—as in their paper—we keep the idea of constant and identical savings rates. Simple trading strategies with constant savings rates are the first set of rules we are able to consider. Later research will have to extend this analysis, for example to Markovian portfolio rules. With general trading strategies, recently, Blume and Easley (2000) and Sandroni (2000) have investigated the case of endogenous savings in a complete market. They show that among all infinite horizon expected utility maximizers those who happen to have rational expectations will eventually dominate the market. This interesting result holds irrespectively of the investors' risk aversion characteristics. The intuition goes that differences in investors expectations determine their savings rates which are essential for the growth of the investors' wealth.

Since we belief that every utility function is debatable we do not want to base portfolio rules on (infinite horizon) utility maximization problems. Instead we take the notion of the portfolio rule as the primitive concept and then investigate the evolution of any finite set of portfolio rules. Hence for our analysis it is not important whether the portfolio rules could possibly be generated by utility maximization. What counts from an evolutionary point of view is not the utility level but the chances of survival; and, indeed, the unique evolutionary stable strategy that we identify cannot be generated by maximization of a stationary utility function over an infinite horizon. In contrast to the Kelly rule, the unique evolutionary stable portfolio strategy does not maximize expected logarithm of its own wealth but it minimizes the expected logarithm of all potential invaders to a monomorphic population in which the incumbent's portfolio strategy itself determines market prices.

The generalization of Blume and Easley (1992) towards incomplete markets brings the evolutionary portfolio theory a bit further to a possible application. Recall from elementary statistics that the sample mean of the realized relative payoffs is an efficient and unbiased estimator of the expected relative payoffs, provided the process of asset payoffs is ergodic. We assume ergodicity. Hence it is very easy to compute our new strategy on real markets. In Hens and Stalder (2001) we compute the performance of the new strategy on SWX 1999 data. It turns out that our new strategy out-performs the SMI. Moreover, in that paper we also consider an artificial market with short lived asset having returns as generated by the SWX 1999 data. It turns out that in competition with strategies from classical finance (mean-variance optimization) and from behavioral finance (prospect theory) the evolutionary finance strategy gains the biggest wealth share within a couple of weeks. Hence even though the theoretical results are derived for idealized returns and in the long run (i.e. they are asymptotic results on P-almost all paths), in contrast to Samuelson's (1979) critique, these results seem also to be relevant for realistic returns even in the medium run.

In the next section we present the economic model which has the mathematical structure of a random dynamical system. Then we define the equilibrium concepts and the stability notions, Section 3. In Section 4 we discuss and generalize Blume and Easley's result in the case of incomplete markets. Section 5 presents our main result which will be proved using a series of propositions that are also of independent interest. In section 6 we analyze the evolutionary fitness of portfolio rules based on mean-variance optimization to study the issue of under-diversified portfolios. Section 7 concludes.

2 The Model

Time is discrete and indexed by t. The possible states of nature are determined in each period in time by the realization of a stochastic process $\xi = (\xi_t)_{t\in\mathbb{Z}}$ with values in some measurable set $(E, \mathcal{E})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the canonical realization of ξ on the path space, i.e. Ω is the sample path space with representative element $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...)$, $\mathcal{F} = \mathcal{E}^{\mathbb{Z}}$ is the corresponding σ-algebra, and $\mathbb{P}$ is the associated probability measure induced by ξ. ω_t denotes the state of nature at time t, and the sequence of observations up to the end of period t is referred to as ω^t. Further let $\mathcal{F}^t = \sigma\{\omega_u \mid u < t\}$ denote the information available at the beginning of period t. By definition, an $\mathcal{F}^t$-measurable random variable can only depend on ω^{t-1}.

There are finitely many investors $i \in \mathcal{I} = \{1, ..., I\}$, $I \geq 1$, endowed with wealth $w_0^i > 0$ at time 0. Assets $A_t^k(\omega) := A^k(\omega_t)$, $k = 1, ..., K$, with $K \geq 2$, live for one period only but are identically re-born in every period. We make the following

Assumption 1 *For all $t \in \mathbb{Z}$, (1) $A_t^k : \Omega \to \mathbb{R}_+$ for all k; (2) for each k there exists a set $\Omega_k \in \mathcal{F}$ with $\mathbb{P}(\Omega_k) > 0$ such that $A_t^k(\omega) > 0$ for all $\omega \in \Omega_k$; and (3) $\sum_{k=1}^K A_t^k : \Omega \to \mathbb{R}_{++}$.*

This assumption ensures that all assets yield non-negative payoffs in all states of nature, each asset yields a strictly positive payoff for a set of states of non-zero measure, and total payoff of all assets is strictly positive.

In each period in time t every investor selects a portfolio $a_t^i = (a_{1,t}^i, ..., a_{K,t}^i)$ with values in $\mathbb{R}_+^K$. a_t^i is an $\mathcal{F}$-measurable random variable. We can make more specific assumptions (and actually do later on) on the measurability of the variables such as all a_t^i being $\mathcal{F}^t$-measurable for each period in time t. Given the portfolio a_t^i at time t, the investor's wealth in period $t+1$ is given by,

$$w_{t+1}^i = \sum_{k=1}^K A_t^k(\omega)\, a_{k,t}^i \tag{1}$$

Letting $\rho_{k,t}$ denote the price of asset k in period t, then—provided that the agent's wealth is positive—his *budget shares* are given by,

$$\lambda_{k,t}^i := \frac{\rho_{k,t}\, a_{k,t}^i}{w_t^i}$$

We define the *trading strategy* of investor i as a sequence of budget shares $\lambda_t^i = (\lambda_{1,t}^i, ..., \lambda_{K,t}^i)_{t \geq 0}$. If all a_t^i and $\rho_{k,t}$ are $\mathcal{F}^t$-measurable for all t, then each budget share λ_t^i is also $\mathcal{F}^t$-measurable.

Assuming that any investor exhausts his budget in all periods in time, i.e. the portfolio is chosen such that $\sum_{k=1}^K \rho_{k,t}\, a_{k,t}^i = w_t^i$ for all $t \geq 0$, every trading strategy λ_t^i takes values in the unit simplex $\Delta^K := \{x \in \mathbb{R}_+^K | \sum_{k=1}^K x_k = 1\}$.

The market-clearing prices are given by,

$$\rho_{k,t} = \frac{1}{\bar{a}_t^k} \sum_{i=1}^I \lambda_{k,t}^i\, w_t^i \tag{2}$$

where $\bar{a}_t^k > 0$, assumed to be $\mathcal{F}$-measurable, is the total supply of asset k at time t.

For the market selection process to be well defined, we need to guarantee that market prices q_t are always positive. A sufficient condition for this is that some trading strategy with positive initial wealth is *completely mixed*, i.e. it has only strictly positive budget shares. Therefore we make the assumption,

Assumption 2 *In every market there is some trading strategy λ with positive initial wealth $w_0 > 0$ that is completely mixed, i.e. $\lambda(\omega) \in \mathrm{int}\Delta^{\mathrm{K}}$ for all $\omega \in \Omega$.*

It is clear from Assumption 1 and equation (1) that any completely mixed trading strategy with positive wealth in a period of time maintains a positive wealth for all future periods.

Taking into account how equilibrium prices are determined, we obtain a recursive formula for the total wealth of each consumer. Consumer i's wealth in period $t+1$ is given by,

$$w^i_{t+1} = \sum_{k=1}^{K} A^k_t(\omega)\,\bar{a}^k_t\;\frac{\lambda^i_{k,t}\,w^i_t}{\sum_{j=1}^{I}\lambda^j_{k,t}\,w^j_t} \tag{3}$$

and the total market wealth in period $t+1$, $w_{t+1} := \sum_i w^i_{t+1}$, can be equated as,

$$w_{t+1} = \sum_{i=1}^{I} w^i_{t+1} = \sum_{k=1}^{K} A^k_t(\omega)\,\bar{a}^k_t \tag{4}$$

From (3) and (4) we obtain a recursive formula for the evolution of the wealth (or market) shares $r^i_t := w^i_t/w_t$,

$$r^i_{t+1} = \sum_{k=1}^{K} \frac{A^k_t(\omega)\,\bar{a}^k_t}{\sum_{l=1}^{K} A^l_t(\omega)\,\bar{a}^l_t}\;\frac{\lambda^i_{k,t}\,r^i_t}{\sum_{j=1}^{I}\lambda^j_{k,t}\,r^j_t} \tag{5}$$

Finally we define the relative payoff of asset k as,

$$R^k_t(\omega) := \frac{A^k_t(\omega)\,\bar{a}^k_t}{\sum_{l=1}^{K} A^l_t(\omega)\,\bar{a}^l_t}$$

Assumption 3 *The supply of each asset is deterministic and independent of time. By appropriate normalization of the expected payoff of each asset we can and do assume $\bar{a}^k_t \equiv 1$ for all k.*

Assumption 3 obviously ensures that the relative payoff of each asset, $R^k_t(\omega)$, also depends only on the state of ω_t. Further, Assumption 3 ensures that market wealth $w_{t+1}(\omega^t) \equiv w(\omega_t)$ is completely determined by the current state of nature.

The prices of the assets normalized by the market wealth are given by,

$$q_{k,t} := \rho_{k,t}/w_t = \sum_{i=1}^{I} \lambda^i_{k,t}\,r^i_t \tag{6}$$

i.e. the normalized price is a convex combination of the trading strategies for asset k over the wealth shares of investors. $q_{k,t}$ is an $\mathcal{F}$-measurable random variable; it is $\mathcal{F}^t$-measurable, if so are all trading strategies.

Note that in the definition of next period wealth we have assumed that no investor saves or withdraws any money along the process. Our results carry over to the case of saving rates which are constant over time and identical among investors.

We will restrict our study of the evolution of wealth shares to the case of trading strategies that are deterministic and constant over time. Following the terminology of Blume and Easley (1992) we define,

Definition 1 *A* simple trading strategy*—also called* portfolio rule*—is a deterministic vector of budget shares that is hold constant over time, i.e.* $\lambda_t^i(\omega) \equiv \lambda^i \in \Delta^K$.

The restriction to simple trading strategies will permit clear-cut results on the success of trading strategies. The main results of Blume and Easley (1992) are also derived under this assumption.[2]

The model considered in Blume and Easley (1992) is derived from the above model by assuming that there are finitely many states of nature and that the state in each period is determined according to an i.i.d. random draw.

The budget shares that an investor with a simple trading strategy allocates to each asset is independent of the observed history and of time. Under this assumption we can re-formulate the process of the evolution of wealth shares (5) for a given set of simple trading strategies $(\lambda^i)_{i\in\mathcal{I}}$. Before doing so, we need to introduce the appropriate framework for our analysis. First, we need an ergodic-theoretic representation of uncertainty. Second, we need a dynamical description of the evolution of wealth shares.

Assumption 4 *The states of nature are determined by an ergodic process.*[3]

The shift map θ on the space of all sample paths Ω is defined by $\theta\omega(\cdot) = \omega(1+\cdot)$. Denote by θ^t the t-times iterate of θ. The family θ^t, $t \in \mathbb{Z}$ defines a measurable flow on Ω, i.e. $\theta^{t+u} = \theta^t \circ \theta^u$ for all $u, t \in \mathbb{Z}$, $\theta^0 = \mathrm{id}_\Omega$, and θ is measurable and measurably invertible. By Assumption 4, θ and θ^{-1} are

[2]In our discussion of the Blume and Easley (1992) result with diagonal securities we allow for general $\mathcal{F}$-measurable trading strategies.

[3]This assumption also discharges us from using the common "almost surely," because any condition or result with this additional restriction can be transferred into a "for all ω" statement by restricting the space Ω to an invariant subset of full $\mathbb{P}$-measure. This claim only holds in that generality because time is discrete here.

ergodic with respect to $\mathbb{P}$. The list $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called an ergodic dynamical system.

For any fixed set of simple strategies $(\lambda^i)_{i\in\mathcal{I}}$ the evolution of wealth shares (5) can be written as,

$$r_{t+1} = f(\theta^t\omega, r_t) \tag{7}$$

where

$$f_i(\theta^t\omega, r) = \sum_{k=1}^{K} R^k(\theta^t\omega)\, \frac{\lambda^i_k\, r^i}{\sum_{j=1}^{I} \lambda^j_k\, r^j} \tag{8}$$

and $R^k(\omega) := R^k_0(\omega)$ (and thus $R^k(\theta^t\omega) = R^k_t(\omega)$). By ergodicity $\mathbb{E}R^k(\theta^t\omega) \equiv \bar{R}^k(\leq 1)$, i.e. the expected relative payoff of each asset is constant over time.

We refer to equation (7) as the *market selection process* in the following.

The dynamical description of the market selection process employs the framework of a *random dynamical system*, Arnold (1998), see also Schenk-Hoppé (2001). (7) generates a random dynamical system in the following sense. Let $f(\omega) := f(\omega, \cdot) : \Delta^I \to \Delta^I$, and define

$$\varphi(t, \omega, r) := \begin{cases} f(\theta^{t-1}\omega) \circ \ldots \circ f(\omega)r & \text{for } t \geq 1 \\ r & \text{for } t = 0 \\ f(\theta^t\omega)^{-1} \circ \ldots \circ f(\theta^{-1}\omega)^{-1} r & \text{for } t \leq -1 \end{cases} \tag{9}$$

In words, $\varphi(t, \omega, r)$ is the vector of wealth shares of all investors at time t when the initial distribution of wealth shares is r and the sequence of realizations of states ω prevails.

$f^{-1}(\omega)$ is well-defined for all $\omega \in \Omega$ if all strategies are completely mixed and there are no redundant assets. In this case φ is defined for all $t \in \mathbb{Z}$; otherwise $t \in \mathbb{N}$. In the following we restrict attention to the case in which f is invertible.

The family of maps $\varphi(t, \omega, r)$ is a *random dynamical system* on the unit simplex Δ^I. That is, $\varphi : \mathbb{Z} \times \Omega \times \Delta^I \to \Delta^I$, $(t, \omega, r) \mapsto \varphi(t, \omega, r)$ is a $\mathcal{B}(\mathbb{Z}) \otimes \mathcal{F} \otimes \mathcal{B}(\Delta^I), \mathcal{B}(\Delta^I)$ measurable[4] mapping such that $\varphi(0, \omega) = \mathrm{id}_{\Delta^I}$ and $\varphi(s + t, \omega) = \varphi(t, \theta^s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{Z}$, and $\omega \in \Omega$.

We refer the reader to the monograph by Arnold (1998) for any additional information.

It is important to emphasize that the random dynamical system generated by (7) depends on the trading strategies pursued by the investors. That is, for any set of simple strategies $(\lambda^i)_{i\in\mathcal{I}}$ there is a unique random dynamical system generated by (7). We will refer to such a random dynamical system as being associated to the set of strategies $(\lambda^i)_{i\in\mathcal{I}}$.

[4] $\mathcal{B}$ denotes the Borel σ-algebra, and $\mathcal{B}(\Delta^I) := \mathcal{B}(\mathbb{R}^I \cap \Delta^I)$ is the trace σ-algebra.

3 Evolutionary Stability

In this section we introduce the stability concepts needed to analyze the long term behavior of the wealth shares under the market selection process (7).

Given a random dynamical system for a set of simple trading strategies $(\lambda^i)_{i\in\mathcal{I}}$, $I > 0$, one is particularly interested in those wealth shares that evolve in a stationary fashion over the infinite time-horizon. Here we restrict ourselves to deterministic distributions of wealth shares that are fixed under the market selection process (7).[5] To specify this notion, we recall the definition of a deterministic fixed point in the framework of random dynamical systems.

Definition 2 *Given a set of strategies $(\lambda^i)_{i\in\mathcal{I}}$ with associated random dynamical system φ, $\bar{r} \in \Delta^I$ is called a (deterministic)* fixed point *of φ if for all $\omega \in \Omega$,*

$$\bar{r} = \varphi(1, \omega, \bar{r}) \ (\equiv f(\omega, \bar{r})) \,. \tag{10}$$

The distribution of wealth shares $\bar{r}$ is said to be invariant *under the market-selection process (7) for the set of strategies $(\lambda^i)_{i\in\mathcal{I}}$.*

Condition (10) is equivalent to $\bar{r} = \varphi(t, \omega, \bar{r})$ for all t and all ω. (See footnote 3 for a justification to use "all ω" instead of "almost surely.")

If $r^i = 0$, then $\varphi^i(t, \omega, r) = 0$ by (7). Therefore, in any set of trading strategies each unit vector in Δ^I is a fixed point, i.e. the state in which one investor possesses the entire market does not change over time.

We are particularly interested in those invariant distributions of wealth shares which are stable under the market selection process. Roughly speaking, stability means that small perturbations of the initial distribution of wealth shares do not have a long-run effect. If an invariant distribution of wealth shares is stable, all sample paths starting in a neighborhood of this distribution at time zero and the sample path of the invariant distribution of the wealth shares are asymptotically identical. We will need different notions of stability; they are defined as follows.

Definition 3 *Given a set of strategies $(\lambda^i)_{i\in\mathcal{I}}$ with associated random dynamical system φ, an invariant distribution of wealth shares $\bar{r} \in \Delta^I$ is called (locally)* stable*, if there exists a random open set $U(\omega)$ containing $\bar{r}$ such that for all ω, $\lim_{t\to\infty} \|\varphi(t, \omega, r) - \bar{r}\| = 0$ for all $r \in U(\omega)$.*

$U(\omega)$ is a random open set, if it is an open set for all ω and $\{\omega \mid (\Delta^I \backslash U(\omega)) \cap G \neq \emptyset\} \in \mathcal{F}$ for all open sets G, cf. Arnold (1998, Chap. 1.6).

[5] See e.g. Schenk-Hoppé (2001) for applications of the general concept of a random fixed point in economic growth.

Given a set of strategies $(\lambda^i)_{i\in\mathcal{I}}$ and a locally stable invariant distribution of wealth shares $\bar{r}$, then any initial distribution of wealth shares in a small neighborhood of $\bar{r}$ is asymptotically identical to $\bar{r}$ as time tends to infinity.

It is straightforward to see that this notion of stability does not make sense on the level of individual investors in general. For example, suppose r is an invariant distribution of wealth shares for the random dynamical system φ associated to the pair of strategies (λ^1, λ^2). Then $\bar{r}_\alpha := (r^1, (1-\alpha)\, r^2, \alpha\, r^2)$ is invariant for the random dynamical system, say φ_α, on Δ^3 associated to $(\lambda^1, \lambda^2, \lambda^2)$ for all $\alpha \in [0, 1]$. However, $\bar{r}_\alpha$ can never (even if r is locally stable for φ) be locally stable for the random dynamical system φ_α on Δ^3.

We will therefore interpret a distribution of wealth shares r as a distribution over populations of players where all players within each group play the same strategy. Thus the wealth share r^i denotes that fraction of the total market wealth belonging to the players of strategy λ^i. Under this assumption it is clear that all strategies λ^i are different from each other, i.e. $\lambda^i \neq \lambda^{i'}$ for all $i, i' \in \mathcal{I}$, $i \neq i'$.

The above definition refers to the stability of a distribution of wealth in a population with given strategies. However, one would also like to have a notion of stability in the case that new strategies occur on the market. We first note that the structure of the market selection process (7) implies the following extension property. Let $(\lambda^i)_{i\in\mathcal{I}}$, $I > 0$, be any set of simple strategies. Suppose $\bar{r}$ is an invariant distribution of wealth shares for the corresponding random dynamical system on Δ^I. Then for any set $(\lambda^j)_{j\in\mathcal{J}}$, $\mathcal{J} = \{1, ..., J\}$ with $J \geq 0$ ($\mathcal{J} = \emptyset$, if $J = 0$), of simple strategies, $(\bar{r}, 0, ..., 0) \in \Delta^{I+J}$ (J-times zero) is an invariant distribution of wealth shares for the random dynamical system on Δ^{I+J} associated to the set of simple strategies $((\lambda^i)_{i\in\mathcal{I}}, (\lambda^j)_{j\in\mathcal{J}})$.

Given a set of strategies $(\lambda^i)_{i\in\mathcal{I}}$ another set of strategies $(\lambda^j)_{j\in\mathcal{J}}$ is called *new*, if $\lambda^j \neq \lambda^i$ for all $j \in \mathcal{J}$, $i \in \mathcal{I}$ and $\lambda^j \neq \lambda^{j'}$ for all $j, j' \in \mathcal{J}$, $j \neq j'$, i.e. adding a set of new strategies yields a market in which no redundant strategies are present.

Definition 4 *Given a set of simple strategies $(\lambda^i)_{i\in\mathcal{I}}$ with associated random dynamical system φ on Δ^I. An invariant distribution of wealth shares $\bar{r} \in \Delta^I$ is called* evolutionary stable, *if for all $J \geq 0$, $(\bar{r}, 0, ..., 0) \in \Delta^{I+J}$ is stable for all sets of strategies $((\lambda^i)_{i\in\mathcal{I}}, (\lambda^j)_{j\in\mathcal{J}})$ with $(\lambda^j)_{j\in\mathcal{J}}$ being new.*

For each evolutionary stable distribution of wealth shares there exits an entry barrier (a random variable here) below which an arbitrary number of new strategies do not drive out the incumbent players. Any perturbation, if sufficiently small, does not change the long-run behavior of the distribution of wealth shares. The market selection process asymptotically leaves the mutants with no wealth share while the market is shared between the incumbents as unchanged. As discussed above, we do not allow for redundant strategies to be introduced.

Unlike in evolutionary game theory, this notion of stability refers to the distribution of wealth shares and not to the set of strategies. It may well be the case that for a given set of strategies there are two different stable invariant distributions of wealth shares one of which being evolutionary stable and the other not.

If there is only one strategy present in the market this strategy is assigned the total wealth. It therefore makes sense to define,

Definition 5 *A strategy is called* evolutionary stable, *if the invariant distribution of wealth shares* $1 \in \Delta^1$ *is evolutionary stable.*

Finally, we define a corresponding local stability criterion.

Definition 6 *Given a set of simple strategies* $(\lambda^i)_{i\in\mathcal{I}}$ *with associated random dynamical system* φ *on* Δ^I. *An invariant distribution of wealth shares* $\bar{r} \in \Delta^I$ *is called* locally evolutionary stable, *if for all* $J \geq 0$ *there exists a constant* $\delta > 0$ *such that* $(\bar{r}, 0, ..., 0) \in \Delta^{I+J}$ *is locally stable for all sets of strategies* $((\lambda^i)_{i\in\mathcal{I}}, (\lambda^j)_{j\in\mathcal{J}})$ *with* $(\lambda^j)_{j\in\mathcal{J}}$ *being new and* $\min_{i\in\mathcal{I}} \max_{j\in\mathcal{J}} \|\lambda^i - \lambda^j\| < \delta$.

A locally evolutionary stable distribution of wealth shares is evolutionary stable with respect to local mutations. That is, the strategies that can be pursued by all mutants are limited to small deviations from existing strategies.

We can now turn to the study of the long-run outcome of the market selection process.

4 Blume and Easley's result revisited

We briefly outline Blume & Easley's (1992) findings in the framework of random dynamical systems theory, extending their result to the case in which the set of all possible states of nature is an arbitrary set and the random draw is *ergodic*.

Analogous to Blume and Easley (1992) we assume that the payoffs of the assets are diagonal. That is, there exists a measurable partition $(\Omega_k)_{k=1,...,K}$ of Ω into sets with strictly positive measure such that $R^k(\omega) > 0$ if and only if $\omega \in \Omega_k$. (In fact $R^k(\omega) \in \{0, 1\}$ for diagonal securities.) Due to this property we can unambiguously denote the relative payoffs in the market selection process (7) at time t by $R(\theta^t\omega)$.

A trading strategy of investor i is an $\mathcal{F}$-measurable random variable $\lambda^i(\omega) \in \Delta^K$. We denote by $\lambda^i_{\theta^t\omega}$ the wealth share invested in that asset k with $\theta^t\omega \in \Omega_k$.

Due to the assumption of diagonal payoffs (7) simplifies to,

$$r^i_{t+1} = R(\theta^t\omega)\,\frac{\lambda^i_{\theta^t\omega}\,r^i_t}{\sum_{j=1}^I \lambda^j_{\theta^t\omega}\,r^j_t}$$

for any sample path ω. The evolution of the ratio of the wealth shares of any two investors, say i and j, can then be written as,

$$\frac{r^i_{t+1}}{r^j_{t+1}} = \frac{\lambda^i_{\theta^t\omega}}{\lambda^j_{\theta^t\omega}}\,\frac{r^i_t}{r^j_t}$$

In other words, the normalized asset price, $\sum_{j=1}^I \lambda^j_{\theta^t\omega}\,r^j_t$, cancels out for diagonal securities.

Fix any deterministic initial wealth shares $r^i_0 > 0$ and $r^j_0 > 0$. Then the asymptotic behavior of the ratio of the two wealth shares is given by,

$$\lim_{T\to\infty}\frac{1}{T}\,\log\frac{r^i_T}{r^j_T} = \lim_{T\to\infty}\frac{1}{T}\,\sum_{t=0}^{T-1}\log\frac{\lambda^i_{\theta^t\omega}}{\lambda^j_{\theta^t\omega}} = \mathbb{E}\log\frac{\lambda^i_\omega}{\lambda^j_\omega}$$

for almost all sample paths ω. The equality on the far right-hand side of the last equation holds by the (Birkhoff–Chintchin) ergodic theorem. The expected value is finite if the strategies are completely mixed.

Consequently we obtain that along almost any sample path ω,

$$\lim_{T\to\infty}\frac{1}{T}\,\log\frac{r^i_T}{r^j_T} > 0 \text{ if and only if } \mathbb{E}\log\lambda^i_\omega > \mathbb{E}\log\lambda^j_\omega \tag{11}$$

The equation on the left-hand side implies that for almost all ω, $\log r^i_T(\omega) \geq T\varepsilon + \ln r^j_T(\omega)$ for all sufficiently large T, where $\varepsilon > 0$. Since $\log r^i_T(\omega) \leq 0$ for all T and all ω, $\ln r^j_T(\omega) \to -\infty$ as $T \to \infty$. Thus, we find that $r^i_T(\omega) \to 1$ almost surely.

This result implies the following asymptotic behavior of the market selection process. Those investors who are closest to maximizing the expected logarithm of the wealth shares will eventually dominate the market. This result holds regardless of the initial distribution of wealth shares in the population. However, note that even in the case discussed here the surviving population depends on the strategies present in the population.

The best choice an investor can make in a period t is to set $\lambda^i_{\theta^t\omega} = 1$ if and only if $\theta^t\omega \in \Omega_k$. However, this requires knowledge of the state ω_t prior to the revelation of the random draw at time t.

If the information available at time t is given by $\mathcal{F}^t$, i.e. $\lambda^i(\theta^t\omega)$ has to be $\mathcal{F}^t$-measurable (and thus can only depend on ω^{t-1} at time t), the optimal portfolio rule depends on the specific stochastic process that determines the state of the world. For instance if the underlying ergodic process stems for a Markov process, then the optimal portfolio rule is one that depends on the last observation ω_{t-1}. Further information from the observed history is not helpful in that case.

Let us turn to the case being closest to Blume and Easley (1992). Suppose the random draw is i.i.d. and that all trading strategies $\lambda^i(\theta^t\omega)$, $i \in \mathcal{I}$, are $\mathcal{F}^t$-measurable. Then the state ω_t is independent of the observed history up to time t, ω^{t-1}. In this case the past does not contain any information on the future and thus it is rational to play a simple trading strategy.

If trading strategies are simple and the state of nature can only take finitely many values, $s = 1, ..., S$ with $\mathbb{P}\{\omega \mid \omega_t = s\} \equiv p_s > 0$, the right-hand side of equation (11) becomes

$$\sum_{s=1}^{S} p_s \log \lambda_s^i > \sum_{s=1}^{S} p_s \log \lambda_s^j$$

Consequently—as in Blume and Easley (1992)—we obtain that those investors who are closest to maximizing the expected logarithm of the wealth shares λ_s (under the distribution of the one-period random draw) will eventually dominate the market. The strategy maximizing the expected logarithm of the budget shares λ_s is "betting your beliefs", i.e. $\lambda_s = p_s$ with $s = 1, ..., S$.

As already pointed out above, if this strategy is present in the population then it is the unique long-run outcome of the market-selection process. In a market in which only simple strategies are present, equation (11) becomes an absolute fitness criterion in the sense that it is independent of the population under consideration. With diagonal securities and an i.i.d. random draw maximizing the expected logarithm means maximizing the growth rate of wealth in *any* population.

5 Main Result

In this section we present our main result. We first characterize the set of stationary wealth shares as being monomorphic populations. That is to say, any population of traders in which every trader uses the same trading strategy gives rise to an invariant distribution of wealth shares and if there are no redundant assets then only such monomorphic populations correspond to stationary solutions. Then we show that only the portfolio rule that we propose is evolutionary stable. Any other portfolio rule can be driven out even by portfolio rules arbitrary close to it, i.e. it is not even locally evolutionary stable. The central mathematical tool is Oseledets's multiplicative ergodic theorem.

With a general payoff matrix we can no longer benefit from the cancellation of prices in the evolution of relative wealth shares and there are some important conceptual differences to the case of diagonal securities. In contrast to that case there is no longer an absolute fitness criterion for the survival of trading strategies. The growth rate of any trading strategy now depends essentially on the population in which it lives. Restricting attention to the question of local stability of deterministic invariant distributions circumvents these problems and is still sufficient to single out a unique trading strategy. Before presenting the main result, we derive two auxiliary results that are also of independent interest.[6]

We have already noted that every distribution of wealth shares in which the players of only one strategy possess the entire market wealth is invariant under the market selection process (and is a deterministic fixed point). Moreover, if there are no redundant assets there is also a converse to this observation as the following result shows.

Assumption 5 *There are no redundant assets.*

Assumption 5 requires that for any two portfolios $a^1, a^2 \in \Delta^K$ with $a^1 \neq a^2$, $\sum_k R^k(\omega)(a^1_k - a^2_k) \neq 0$ on a set of strictly positive measure. That is different portfolios cannot generate the same stream of dividends.

Proposition 1 *Under the maintained assumptions only one strategy can have strictly positive wealth in every population of strategies with a (deterministic) invariant distribution of wealth shares.*

The proof is relegated to the Appendix.

In Proposition 1 all deterministic invariant distributions of wealth shares are characterized. We next derive a sufficient condition for the stability of such fixed points. The following Proposition is the central auxiliary result of the main Theorem.

Proposition 2 *Given a set of strategies $(\lambda^i)_{i\in\mathcal{I}}$. Under the maintained assumptions the invariant distribution of wealth shares $\bar{r} = e_n$ being concentrated on the players of the n-th strategy (which is completely mixed by Assumption 2) is*

[6] Güth in (Güth and Ludwig 2000) has also studied the issues addressed in the two following propositions for the case of a finite state space.

(i) stable, if

$$\mathbb{E}\log\left(\sum_{k=1}^{K} R^k(\omega)\,\frac{\lambda_k^i}{\lambda_k^n}\right) < 0 \qquad \textit{for all } i \neq n; \tag{12}$$

(ii) unstable, if for some $i \neq n$

$$\mathbb{E}\log\left(\sum_{k=1}^{K} R^k(\omega)\,\frac{\lambda_k^i}{\lambda_k^n}\right) > 0. \tag{13}$$

The proof is given in the Appendix.

The results in Proposition 2 have the following interpretation. In a situation in which the prices of all assets are determined by the portfolio rule λ^n we can measure the exponential growth rate of other, competing portfolio rules. If the invariant distribution of wealth shares $\bar{r} = e_n$ is stable then the strategy λ^n which fixes the prices has a higher growth rate in a neighborhood of this distribution of market wealth than all other portfolio rules in the population. However, if there is at least one strategy that has a higher growth rate for these prices, $\bar{r} = e_n$ is unstable and the λ^n-player does not reobtain total market wealth after a slight deviation from the possing-everything situation.

The main result of our paper is based on the observation that, allowing for all possible mutations, only one particular strategy satisfies the necessary condition for stability derived in Proposition 2.

Theorem 1 *Under the maintained assumptions, the simple strategy $\lambda^\star$ defined by,*

$$\lambda_k^\star = \mathbb{E}R^k(\omega),$$

for $k = 1, ..., K$ is evolutionary stable, and no other strategy is locally evolutionary stable.

Again the proof can be found in the Appendix.

The portfolio rule $\lambda^\star$ divides wealth according to the expected relative payoffs of the assets. For a given asset market of the structure discussed in this paper, the strategy is very simple to compute; it requires a minimum of easily accessible information.

Let us consider the case in which the state of nature can only take finitely many values $s = 1, ..., S$ in detail. Under the assumption of ergodicity $\mathbb{P}\{\omega \mid \omega_t = s\} \equiv p_s > 0$ for all s. Therefore, the portfolio rule $\lambda^\star$ in Theorem 1 becomes $\lambda_s^\star = \sum_{s=1}^{S} p_s R^k(s)$. It is straightforward to see that we reobtain the result by Blume and Easley (1992, Section 3) in the case of diagonal securities. In this case, $\lambda^\star$ corresponds to the Kelly rule of "betting one's beliefs."

We next relate our result in Theorem 1 to the standard concept of evolutionary game theory. In the following it is shown that the strategy $\lambda^\star$ can be interpreted as a Nash equilibrium.

For this purpose we recall the definition of the auxiliary function (15) which is used in the proof of Theorem 1,

$$\hat{g}(\alpha, \beta) := g_\beta(\alpha) = \mathbb{E} \log \left(\sum_{k=1}^{K} R^k(\omega) \frac{\alpha_k}{\beta_k} \right)$$

$\hat{g}(\alpha, \beta)$ measures the asymptotic exponential growth rate of a strategy α in a population in which all asset prices are determined by strategy β. Using Proposition 2, the assertion of Theorem 1 can be stated as:

For all $\alpha \neq \lambda^\star$,

$$\begin{aligned} \hat{g}(\lambda^\star, \lambda^\star) &> \hat{g}(\alpha, \lambda^\star) \quad \text{and;} \\ \hat{g}(\alpha, \alpha) &< \hat{g}(\beta, \alpha) \quad \text{for some } \beta \text{ in every neighborhood of } \alpha. \end{aligned}$$

That is to say $\lambda^\star$ is the unique symmetric Nash equilibrium in a game with payoff function $\hat{g}$. Moreover, $\lambda^\star$ is also a *strict* Nash equilibrium. Therefore $\lambda^\star$ is the unique evolutionary stable strategy in the sense of Maynard Smith and Price (1973), i.e. the population in which all investors play $\lambda^\star$ is resistant to the invasion of any new portfolio rule.

6 Mean-Variance Optimization

In this section we analyze the evolutionary fitness of portfolio rules based on mean-variance optimization. According to the CAPM, mean-variance optimization leads to a well diversified portfolio, the market portfolio, provided every investor bases his portfolio choice on the mean-variance principle—an assumption that is clearly not true in practice. It is also well known that in practice mean-variance portfolios are often under diversified, i.e. they typically put positive weight on very few assets only. To cure this defect it is then usually suggested to modify the mean-variance portfolio by devoting some positive but small share of the budget on every asset in the portfolio, ensuring that the portfolio is completely mixed. The next result shows that this commonly used "quick fix" of the under-diversification problem is indeed an improvement of the mean-variance portfolio.

Corollary 1 *Suppose $\hat{\lambda}$ is an under-diversified simple strategy, i.e. $\hat{\lambda}_k = 0$ for at least one k. Denote by $\hat{\lambda}^\varepsilon_k := (1-\varepsilon)\hat{\lambda}_k + \varepsilon/S$, $0 < \varepsilon \leq 1$, the corresponding ε-completed strategy. Then $\hat{\lambda}^\varepsilon$ is robust against $\hat{\lambda}$-mutants for all sufficiently small $\varepsilon > 0$, i.e. the distribution of wealth shares that assigns total wealth to the $\hat{\lambda}^\varepsilon$-player is stable in the population $(\hat{\lambda}^\varepsilon, \hat{\lambda})$.*

See the Appendix for a proof.

Even though using the "quick fix" to prevent under-diversification is better than investing according to the under-diversified portfolio rule, it is clear from the main result Theorem 1, that ε-completed under-diversified simple strategies are not locally stable (if they do not coincide with $\lambda^\star$). However, we next show that the situation for ε-completed portfolio rules $\hat{\lambda}^\varepsilon$ is even worse. Any completely mixed simple strategy drives out $\hat{\lambda}^\varepsilon$ for all small enough $\varepsilon > 0$.

Corollary 2 *Given any completely mixed simple strategy λ^c and any under-diversified simple strategy $\hat{\lambda}$. Then $\hat{\lambda}^\varepsilon$, defined in Corollary 1, is not robust against λ^c-mutants for all sufficiently small $\varepsilon > 0$, i.e. the distribution of wealth shares that assigns total wealth to the $\hat{\lambda}^\varepsilon$-player is not stable in the population $(\hat{\lambda}^\varepsilon, \lambda^c)$.*

The proof of this corollary is given in the Appendix.

7 Conclusions

In this paper we have suggested an evolutionary portfolio theory which is based on an evolutionary process of market selection and mutations of simple trading strategies. We identify a portfolio rule as the unique evolutionary stable strategy in a possibly incomplete market of short-lived assets. The strategy divides wealth according to the expected relative payoffs of the assets. It is very simple to compute and (under the assumption of ergodicity of the payoffs) the information needed to implement the strategy is easily available. The success of the strategy characterized in this paper is based on two principles: frequent repetition of an elementary situation and evolutionary market interaction. Using the first principle well-known results on the convergence of stochastic dynamical systems (such as the multiplicative ergodic theorem) can be applied to gain structure for an otherwise intricate problem. The second principle emphasizes that portfolio choice is not only a matter of individual optimization but also of strategic interaction in a competition for market capital. As we have seen the strategy we derive here is the only strategy that is fit enough to drive out any other simple strategies.

We consider the results of this paper as a first step toward a rigorous evolutionary theory of portfolio selection. Further research has to be done for the case of more general trading strategies, such as adapted or Markovian strategies. Moreover, the model should be extended to long-lived assets. The latter extension is extremely important because capital gains are often more relevant for portfolio decisions than dividend payments. We view the assumption of short-lived assets as the major obstacle for real-world applications of evolutionary portfolio theory.

A Appendix

Proof of Proposition 1. We prove the statement by contraposition. Let λ^i, $i \in \mathcal{I}$, be a family of simple trading strategies such that $\lambda^i \neq \lambda^j$ for some $i, j \in \mathcal{I}$, $i \neq j$. Let $r \in \Delta^I$ with $r^i r^j > 0$. We will show that r cannot be invariant.

Since $\lambda^i, \lambda^j \in \text{int}\Delta^{\text{K}}$, and $r^i r^j > 0$, $\lambda^i r^i \neq \lambda^j r^j$. This further implies that $a^i := (\lambda^i_k r^i / \sum_{l=1}^I \lambda^j_k r^l)_{k=1,\dots,K} \neq (\lambda^j_k r^j / \sum_{l=1}^I \lambda^j_k r^l)_{k=1,\dots,K} =: a^j$, i.e. the 'portfolios' a^i and a^j are different. Due to the non-redundancy Assumption 5, $f_i(\omega, r) \neq f_j(\omega, r)$ in equation (7) on a set $\tilde{\Omega} \subset \Omega$ of strictly positive measure. Hence r is not invariant in the sense of Definition 2. $\square$

Proof of Proposition 2. The proof is mainly an application of Oseledets's multiplicative ergodic theorem for random dynamical systems on manifolds, see Arnold (1998, Chapter 4).

The random dynamical system describing the evolution of wealth shares is defined on the simplex Δ^I, an $I-1$-dimensional manifold with boundary. We therefore transform the system and consider a conjugate random dynamical system on a subset of the Euclidean space.

Define the projection of the unit simplex,

$$D^{I-1} := \left\{ y \in \mathbb{R}^{I-1} \mid y_i \geq 0, \sum_{i=1}^{I-1} y_i \leq 1 \right\} \subset \mathbb{R}^{I-1}_+.$$

Further, for each $n \in \mathcal{I}$ define the map

$$h_n : D^{I-1} \to \Delta^I, \; h_n(y_1, ..., y_{I-1}) := \left(y_1, ..., y_{n-1}, 1 - \sum_{i=1}^{I-1} y_i, y_n, ..., y_{I-1} \right).$$

with inverse

$$h_n^{-1}(x_1, ..., x_I) := (x_1, ..., x_{n-1}, x_{n+1}, ..., x_I).$$

h_n is a C^∞-diffeomorphism. We obtain the conjugate random dynamical system on D^{I-1},

$$\psi_n(t,\omega) := h_n^{-1} \circ \phi(t,\omega) \circ h_n.$$

Due to the definition of the space D^{I-1}, we can take directional derivatives in the direction of all unit vectors at all points in the interior of D^{I-1} relative to $\mathbb{R}_+^{I-1}$. That is we can determine the Jacobian of the conjugate system at all points in $\{y \in \mathbb{R}_+^{I-1} \mid \sum_{i=1}^{I-1} y_i < 1\}$ (which is an invariant set for the random dynamical system ψ_n).

Note that the origin of $\mathbb{R}^{I-1}$ corresponds to the nth edge of the unit simplex. The stability properties of these two fixed points are identical because of the C^∞-equivalence of both random dynamical systems.

For notational simplicity we assume without loss of generality that $n = I$. Then the partial derivatives of $\psi_I(1,\omega,y) = h_I^{-1} \circ \phi(1,\omega) \circ h_I(y)$ are given by,

$$\frac{\partial \psi_I^i(1,\omega,y)}{\partial y_m} = -\sum_{k=1}^{K} \frac{R^k(\omega)\,(\lambda_k^m - \lambda_k^I)\,\lambda_k^i\,y_i}{(\sum_{j=1}^{I-1} \lambda_k^j\,y_j + \lambda_k^I\,(1 - \sum_{j=1}^{I-1} y_j))^2}$$

for all $i \neq m$, and by

$$\begin{aligned} \frac{\partial \psi_I^i(1,\omega,y)}{\partial y_m} &= -\sum_{k=1}^{K} \frac{R^k(\omega)\,(\lambda_k^m - \lambda_k^I)\,\lambda_k^m\,y_m}{(\sum_{j=1}^{I-1} \lambda_k^j\,y_j + \lambda_k^I\,(1 - \sum_{j=1}^{I-1} y_j))^2} \\ &\quad + \sum_{k=1}^{K} \frac{R^k(\omega)\,\lambda_k^m}{\sum_{j=1}^{I-1} \lambda_k^j\,y_j + \lambda_k^I\,(1 - \sum_{j=1}^{I-1} y_j)} \end{aligned}$$

for all $i = m$.

The stability properties of $e_I \in \Delta^I$ can be determined by evaluating the Jacobian of ψ_I at the origin and applying the multiplicative ergodic theorem of Oseledets. It will be shown that condition (12) resp. (13) ensures that the top Lyapunov exponent of this linear system is strictly negative resp. positive. Results by Wanner (1995), see Arnold (1998, Theorem 7.5.6), ensure that the dynamic behavior of the linearized system carries over (locally) to the nonlinear stochastic system.

From the above expressions, we obtain the Jacobian of ψ_I at $y = (0, ..., 0)$. It is a diagonal matrix with entry,

$$A_{m,m}(\omega) := \sum_{k=1}^{K} R^k(\omega)\,\frac{\lambda_k^m}{\lambda_k^I}$$

The multiplicative ergodic theorem, Arnold (1998, Theorem 4.2.6), implies that the Lyapunov exponents of the fixed point $y = 0$ of ψ_I are given by

$\lim_{T\to\infty} \frac{1}{T} \log|\prod_{t=0}^{T} A_{m,m}(\theta^t\omega)|$, $m = 1, ..., I-1$. The integrability condition of the multiplicative ergodic theorem is satisfied because the space Δ^I is compact. By the (Birkhoff–Chintchin) ergodic theorem, we find that this limit is equal to

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \log \left| \sum_{k=1}^{K} R^k(\theta^t\omega) \, \frac{\lambda_k^m}{\lambda_k^I} \right| = \mathbb{E} \log \left| \sum_{k=1}^{K} R^k(\omega) \, \frac{\lambda_k^m}{\lambda_k^I} \right| \tag{14}$$

Zero is a stable fixed point of ψ_I if the term in (14) is strictly negative for all $m = 1, ..., I-1$. If (14) is strictly positive for some m, then zero is locally unstable. Due to the diagonal structure of the Jacobian, the eigenspaces correspond to the linear spaces spanned by the unit vectors (restricted to the positive orthant $\mathbb{R}_+^{I-1}$).

The stability of the original system on Δ^I at the fixed point $e_I \in \Delta^I$ is determined by the Lyapunov exponents (14). The corresponding eigenspaces are given by the vertices. All summands in (14) are positive and we thus have obtained conditions (12) and (13) of the proposition. $\square$

Proof of Theorem 1. Obviously, $\lambda^\star$ is a completely mixed strategy, i.e. $\sum_{k=1}^{K} \lambda_k^\star = 1$ and $\lambda_k^\star > 0$ for all k. Next we define the auxiliary function,

$$g_\beta(\alpha) := \mathbb{E} \log \left(\sum_{k=1}^{K} R^k(\omega) \frac{\alpha_k}{\beta_k} \right) \tag{15}$$

in accordance with Proposition 2. For each fixed strategy $\beta \in \mathrm{int}\Delta^{\mathrm{K}} \subset \mathbb{R}^{\mathrm{K}}$, $g_\beta : \mathrm{int}\Delta^{\mathrm{K}} \to \mathbb{R}$. $g_\beta(\alpha)$ is the Lyapunov exponent of the distribution of wealth that assigns total wealth to the 'status quo' population that plays strategy β in a market in which α is the only the alternative strategy.

By Proposition 2 the first assertion of the theorem follows if we can show that $g_{\lambda^\star}(\alpha) < 0$ for all $\alpha \in \mathrm{int}\Delta^{\mathrm{K}}$ with $\alpha \neq \lambda^\star$.

We prove that $g_\beta(\alpha)$ is strictly concave for all $\beta \in \mathrm{int}\Delta^{\mathrm{K}}$ and that $g_{\lambda^\star}(\alpha)$ takes its maximum value at $\alpha = \lambda^\star$.

To ensure strict concavity it suffices to show that $\alpha \mapsto g_\beta(\alpha)$ is strictly concave on the space $\mathbb{R}_{++}^K$, because restriction of the domain to the linear subspace $\mathrm{int}\Delta^{\mathrm{K}}$ preserves strict concavity. The function $\log \sum_{k=1}^{K} (R^k(\omega)\, \alpha_k/\beta_k)$ is concave for all ω and—due to the no-redundancy Assumption 5—strictly concave on a set of positive measure. Therefore $g_\beta(\alpha)$ is strictly concave for each fixed $\beta \in \mathrm{int}\Delta^{\mathrm{K}}$.

We can now employ that $\lambda^\star$ is the unique maximum of $g_{\lambda^\star}(\alpha)$ on $\mathrm{int}\Delta^{\mathrm{K}}$ if all directional derivatives at this point are zero.

The partial derivative of $g_\beta(\alpha)$ with respect to the i-th component α_i is given by

$$\frac{\partial g_\beta(\alpha)}{\partial \alpha_i} = \mathbb{E}\frac{R^i(\omega)/\beta_i}{\sum_{k=1}^K R^k(\omega)\frac{\alpha_k}{\beta_k}}$$

Observe that interchanging integration and differentiation is allowed because $\log(\sum_{k=1}^K R^k(\omega)\,\alpha_k/\beta_k)$ is integrable for each fixed α (follows from $\mathbb{E}R^k(\omega) \leq 1 < \infty$ for all k) and $\mathbb{E}(R^i(\omega)/\sum_{k=1}^K R^k(\omega)) \leq 1 < \infty$ (this follows from the fact that $R^k(\omega) \geq 0$ for all k and all ω by assumption). The last equation implies

$$\frac{\partial g_{\lambda^\star}(\lambda^\star)}{\partial \alpha_i} = \mathbb{E}\frac{R^i(\omega)}{\lambda^\star_i} = \mathbb{E}\frac{R^i(\omega)}{\mathbb{E}R^i} \equiv 1$$

for all $i = 1, ..., K$, since $\sum_{k=1}^K R^k(\omega) \equiv 1$ for all ω.

The directional derivative of $g_{\lambda^\star}$ in the direction $(d\alpha_1, ..., d\alpha_K)$ with the restriction $\sum_{k=1}^K d\alpha_k = 0$ (which is a vector in the simplex) is equated as

$$\sum_{i=1}^K \frac{\partial g_{\lambda^\star}(\lambda^\star)}{\partial \alpha_i}\, d\alpha_i = 0.$$

Let us next prove that any simple strategy $\beta \neq \lambda^\star$ with $\beta \in \mathrm{int}\Delta^{\mathrm{K}}$ is not locally evolutionary stable. That is, for any neighborhood of β there exists an α such that $g_\beta(\alpha) > 0$. It suffices to show that the directional derivative of g_β at β is strictly positive in one direction.

Since $\beta \neq \lambda^\star$ and both are points in the simplex there exists $i \neq j$ with $\beta_i > \lambda^\star_i$ and $\beta_j < \lambda^\star_j$. Note that we have assumed a minimum of two assets.

The directional derivative of g_β at β in the direction $d\alpha$ given by $d\alpha_i = -1/2$, $d\alpha_j = 1/2$, and zero otherwise, is given by,

$$\sum_{k=1}^K \frac{\partial g_\beta(\beta)}{\partial \alpha_k}\, d\alpha_k = \sum_{k=1}^K \frac{\mathbb{E}R^k}{\beta_k}\, d\alpha_k = \frac{1}{2}\left(\frac{\lambda^\star_j}{\beta_j} - \frac{\lambda^\star_i}{\beta_i}\right) > 0.$$

$\square$

Proof of Corollary 1. According to Proposition 2(i) it suffices to show that

$$\mathbb{E}\log\left(\sum_{k:\hat{\lambda}_k>0}^K R^k(\omega)\,\frac{\hat{\lambda}_k}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S}\right) < 0$$

for all small $\varepsilon > 0$. The left-hand side of this equation is strictly increased by omitting ε/S in the denominator. We thus obtain the sufficient condition,

$$\mathbb{E}\log\left(\sum_{k:\hat{\lambda}_k>0} R^k(\omega)\right) \leq \log(1-\varepsilon) \tag{16}$$

Since there is at least one k such that $\hat{\lambda}_k = 0$, we find that $\sum_{k:\hat{\lambda}_k>0} R^k(\omega) < 1$ on a set of positive measure (the term is bounded by 1 for all ω), the left-hand side of (16) $\mathbb{E} \log \left(\sum_{k:\hat{\lambda}_k>0} R^k(\omega) \right) < 0$. Therefore (16) is satisfied for all small enough ε. $\square$

Proof of Corollary 2. Again we employ Proposition 2. The local instability result says that the assertion of the Corollary is true, if

$$\mathbb{E} \log \left(\sum_{k=1}^{K} R^k(\omega) \, \frac{\lambda_k^c}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} \right) > 0 \tag{17}$$

for all small $\varepsilon > 0$.

Noting that

$$\sum_{k=1}^{K} \frac{R^k(\omega)\, \lambda_k^c}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} = \sum_{k:\hat{\lambda}_k>0} \frac{R^k(\omega)\, \lambda_k^c}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} + \sum_{k:\hat{\lambda}_k=0} R^k(\omega) \, \frac{S\, \lambda_k^c}{\varepsilon}$$

$\lambda_k^c > 0$ for all k, and $R^k(\omega) > 0$ on a set of positive measure for all k with $\hat{\lambda}_k = 0$, we find that the left-hand side of (17) tends to infinity as $\varepsilon \to 0$. $\square$

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