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Real Financial Interaction: Integrating Supply Side Wage-Price Dynamics and the Stock Market

by

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Real-Financial Interaction:
Integrating Supply Side Wage-Price Dynamics
and the Stock Market

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Abstract

The paper puts forward a macrodynamic model of the real-financial interaction. Regarding the financial sector it focuses on the stock market dynamics, for the real sector it details goods market disequilibrium and two Phillips curves for prices as well as wages. The central link between the two sectors is constituted by Tobin's (average) q. After highlighting the main feedback mechanisms in the real and financial subdynamics, the long-run equilibrium of the integrated 7th-order dynamic system is shown to be locally stable if certain adjustments are sufficiently sluggish, while large values of some reaction parameters can destabilize the economy. Lastly, the analysis reveals the potential for cyclical motion.

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1 Introduction

The present paper can be seen as linking two strands of macroeconomic theory. On the one hand, it emerges from the work of Chiarella and Flaschel (2000) or Chiarella et al. (2000) and thereby incorporates various feedback effects in the real sector that are associated, primarily, with the wage-price dynamics in an inflationary environment. On the other hand, the paper reconsiders an influential model by Blanchard (1981) and Blanchard and Fischer (1989, Ch. 10.4) that combines a dynamic Keynesian multiplier with a richer financial sector. We will, however, modify Blanchard's treatment by relaxing his hypothesis of rational expectations and perfect substitutability of the non-money financial assets. The main contribution of the paper is to build a model of the real-financial interaction that consistently merges these two approaches from the literature. The analysis can furthermore uncover its basic dynamic properties and feedback mechanisms.

The core of the financial-real interaction is constituted by Tobin's q. It is here assumed that investment I varies, not with the real rate of interest, but with Tobin's average q.[1] Regarding the financial sector, it is consequently the stock price dynamics that feeds back on the real sector. The influence of the (nominal) interest rate, i, is in this respect more indirect; it is only used by investors on the stock market, who compare the (inflation-augmented) equity rate of return to it. This difference, then, drives equity prices and ultimately Tobin's q. Regarding the real sector, output Y impacts on the interest rate through the usual transaction motive of money demand. In addition, Y, i.e. capacity utilization, affects the equity rate of return *via* the rate of profit, r, and the corresponding dividend payments of firms. The real-financial interaction to be studied may thus be concisely summarized by the feedback loop $Y \rightarrow (r,i) \rightarrow q \rightarrow I \rightarrow Y$.

This picture still leaves out the role of inflation and the wage-price dynamics. The relative adjustments of wages and prices, into which also inflationary expectations enter, determine income distribution in form of the wage share and, partly, the rate of profit. Due to differential savings of workers and asset owners, the wage share has an impact on aggregate demand *via* consumption expenditures. Its impact on investment is channelled through Tobin's q, since the profit rate is not a direct argument in the investment function but only co-determines the equity rate of return, which in turn has a bearing on the movements of equity prices and Tobin's q.

The points briefly sketched out above already indicate that the economic variables interact in ways that, in total, go beyond what is investigated in low-order macrodynamics of two or three

[1] While economic theory focusses almost exclusively on marginal q as determining investment, the validity of this approach rests on very stringent assumptions concerning markets and adjustment costs. The use of Tobin's average q, by contrast, as *one* of the main explanatory variables of investment remains valid under a wide variety of scenarios. This point is stressed by Caballero and Leahy (1996).

dimensions. In fact, the dynamic system at which we finally arrive will be of dimension 7. We will nevertheless be able to derive a set of meaningful conditions on the behavioural parameters that ensure local stability of the long-run equilibrium position. Elementary destabilizing mechanisms can be recognized, too. Apart from that, there is broad scope for the economy to exhibit cyclical behaviour.

The paper is organized as follows. Section 2 presents the building blocks of the model. Section 3 translates these equations into intensive form, which yields a 7th-order differential equation system. Section 4 studies two-dimensional subdynamics in the real and financial sector, respectively, thus highlighting the main positive and negative feedback loops. Section 5 returns to the full economy and examines the stability and instability conditions of the system's Jacobian matrix. The method of proof that is here described may also be of more general interest beyond this paper's specific model. Section 6 concludes.

2 Formulation of the model

In this section the model is set up in extensive form, which still refers to the level variables. Much of the notation we adopt is standard, otherwise the symbols are introduced as the discussion of the modelling equations goes along.

Constant growth rates

We begin by postulating constant growth rates for labour productivity $z = Y/L$, labour supply L^s, and money M:

$$\hat{z} = g_z = \text{const} \tag{1}$$
$$\hat{L}^s = g_\ell = \text{const} \tag{2}$$
$$\hat{M} = g_m = \text{const} \tag{3}$$
$$g^o = g_z + g_\ell \tag{4}$$
$$\pi^o = g_m - g^o \tag{5}$$

g^o is the real growth rate that must prevail in a long-run equilibrium position, π^o is the rate of inflation supporting this growth path.

When at a later stage of research the model is to be employed to study business cycle mechanisms, the technological assumption on the labour inputs can be augmented by allowing for procyclical variations of labour productivity (relative to some trend), such as this was proposed in Franke (2001). The local stability analysis, however, would not be seriously impaired by this feature, so we may presently proceed with eq. (1) as it stands.

The goods market

The following equations specify the components of final demand Y^d: consumption C, net investment I, replacement investment δK (δ the constant rate of depreciation), and government spending G. They are complemented by a simple tax rule and the laws of motions for capital and output.

$$
\begin{aligned}
Y^d &= C + I + \delta K + G & (6)\\
pC &= wL + (1-s_c)(pY - wL - \delta pK + iB - T_c) & (7)\\
I &= [g^o + \beta_I(q-1)]\,K & (8)\\
G &= \gamma K & (9)\\
T_c &= \theta pK + iB & (10)\\
\dot{K} &= I & (11)\\
\dot{Y} &= g^o(Y - g^o\beta_n y^n K) + \beta_y[Y^d - (Y - g^o\beta_n y^n K)] + g^o\beta_n y^n I & (12)
\end{aligned}
$$

Eq. (7) states that wage income wL is fully spent on consumption, whereas asset owners save a constant fraction s_c of their non-wage income ($0 < s_c \leq 1$). The latter is made up of firms' profits, which are in their entirety distributed to share holders, and interest payments at interest rate i for the presently outstanding fixed-price bonds B (the bond price being normalized at unity), minus (nominal) taxes T_c. As mentioned in the introduction, the investment function (8) focuses on the influence of Tobin's (average) q, which is defined as $q = p_e E/pK$ (E the number of shares and p_e their price). As it is formulated, (8) implicitly presupposes that $q=1$ is indeed supported as the economy's long-run equilibrium value of Tobin's q. This will have to be confirmed later on.[2] Eq. (9) may be seen as a variant of neutral fiscal policy; γ is a constant. The tax rule (10) is conveniently aimed at making the term $iB - T_c$ in the expression for disposable income in (7) likewise proportional to the capital stock.

While the change in fixed capital in (11) is just the capacity effect of investment, eq. (12) for the change in output is a behavioural equation that requires some additional explanation. It arises from the implications of allowing for goods market disequilibrium. This means that a positive excess demand is served from the existing stock of inventories, and production in excess of demand fills up inventories. To keep the model simple we here neglect possible dynamic feedback effects from these inventories. Nevertheless, since the economy is growing over time, inventories even if remaining in the background must be growing, too. Accordingly, besides producing Y^{md} to meet final demand, firms produce an additional amount to increase a stock of desired inventories N^d. Adopting the equilibrium growth rate g^o for this purpose, we have $Y = Y^{md} + g^o N^d$.

[2] The stability analysis would not be affected if also a (limited) capacity utilization effect on I is considered in (8), which could be easily represented by the output-capital ratio.

The output component Y^{md} is adjusted to reduce the current gap between final demand Y^d and Y^{md}. Letting β_y be the corresponding adjustment speed and taking (trend) growth into account, Y^{md} evolves like $\dot{Y}^{md} = g^o Y^{md} + \beta_y (Y^d - Y^{md})$. On the other hand, desired inventories N^d may be proportional with factor β_n to productive capacity Y^n, which in turn is linked to the capital stock K by what may be called a 'normal' output-capital ratio y^n. Hence, $N^d = \beta_n Y^n = \beta_n y^n K$. y^n itself is treated as an exogenous and constant technological coefficient. Differentiating the thus determined output Y with respect to time, then, yields eq. (12).

Wage-price dynamics

The adjustment of wages brings the employment rate λ into play, which is defined in(13). Labour demand and supply, L and L^s, are measured in hours, and L^s is thought to refer to normal working hours. Hence λ may well exceed unity, if workers work overtime or firms organize extra shifts. $\lambda = 1$ corresponds to normal employment.

$$\lambda = L \,/\, L^s \tag{13}$$
$$\hat{p} = \kappa_p(\hat{w} - g_z) + (1-\kappa_p)\pi + \beta_p(y - y^n) \tag{14}$$
$$\hat{w} = g_z + \kappa_w \hat{p} + (1-\kappa_w)\pi + \beta_w(\lambda - 1) \tag{15}$$
$$\dot{\pi} = \beta_{\pi p}\,(\hat{p} - \pi) \tag{16}$$

The subsequent two equations (14) and (15) put the determination of prices and wages on an equal footing by positing a price as well as a wage Phillips curve, augmented by inflationary expectations π. In their core, both price and wage inflation respond to the pressure of demand on the respective markets. These are the deviations of capacity utilization from normal on the one hand (as they are captured by the difference between the output-capital ratio $y = Y/K$ and the normal ratio y^n), and the deviations of the employment rate from normal on the other hand. For price inflation, the cost-push term is a weighted average of expected inflation, π, and current wage inflation (corrected for labour productivity growth); in parallel, the cost-push term for wage inflation is a weighted average of the same rate of expected inflation and current price inflation. It goes without saying that the two weighting coefficients κ_p and κ_w are between zero and one.[3]

In eq. (16), the rate of expected inflation is supposed to be governed by adaptive expectations with adjustment speed $\beta_{\pi p}$. As π refers, not to the next (infinitesimally) short period, but to a longer time horizon of about a year, say, this rate may perhaps be better called a general

[3] Franke (2001) points out that, in their combination, the Phillips curves (14) and (15) have undesirable cyclical implications. It is shown there that the cyclical features can be improved upon by adding further adjustment mechanisms in the two equations, which are both channelled through the wage share. This idea may be taken up in a later version of the model.

inflation climate. With this interpretation, it also makes more sense that, as implied by (16), π systematically lags behind $\hat{p}$. Incidentally, a similar pattern is found in the survey forecasts made in the real world (see, e.g., Evans and Wachtel, 1993, fig. 1 on p. 477, and pp. 481ff).

The money market

The bond rate of interest i is determined by an ordinary LM schedule. With parameters $\beta_{mo}, \beta_{mi} > 0$, it is formulated as

$$M \;=\; pY\,(\beta_{mo} \,-\, \beta_{mi}\, i) \tag{17}$$

The stock market dynamics

The third financial asset besides money and government bonds are equities. In eq. (18) it is explicitly stated that they are issued by firms to finance investment (and there is no other source of internal or external finance). The next two equations put forward an elementary speculative dynamics on the stock market in two variables: the equity price p_e and expected capital gains π_e, i.e., the expected rate of stock price inflation. Eq. (21) defines the real rate of return, r, in the real sector.

$$pI \;=\; p_e \dot{E} \tag{18}$$

$$\hat{p}_e \;=\; \beta_e\,(r\,pK/p_eE + \pi_e - \xi - i) \;+\; \kappa_e\pi_e + (1-\kappa_e)\pi^o \tag{19}$$

$$\dot{\pi}_e \;=\; \beta_{\pi e}\,(\hat{p}_e - \pi_e) \tag{20}$$

$$r \;=\; (pY - wL - \delta\, pK)\,/\,pK \tag{21}$$

The adjustment equation (19) rests on the supposition that, unlike in the usual LM treatment, bonds and equities are imperfect substitutes. While perfect substitutability between bonds and equities conveys the notion that any difference in the rates of return of the two assets would immediately be eliminated by arbitrage sales or purchases and the corresponding price changes, imperfect substitutability implies that these price adjustments take place at a finite speed. The (nominal) rate of return on equities is given by the dividends which, as has already been mentioned above, are fully paid out to the share holders. In addition, the expected capital gains π_e have to be considered, and a risk premium ξ may be deducted. Thus, $r_e := rpK/p_eE + \pi_e - \xi$ is compared to the bond rate i. $r_e > i$ means that equities have become more attractive than the alternative asset, the consequence being that the demand for equities increases and bids up the equity price; downward adjustments of p_e take place if $r_e < i$.

Eq. (19) expresses this mechanism in growth rate form. The equation furthermore takes into account that the changes in stock prices have to be related to a general growth trend of p_e.

This is specified as a weighted average of currently expected stock price inflation π_e and inflation of stock prices in a long-run equilibrium, which as shown below must be equal to the long-run equilibrium rate of price inflation π^o.

A low value of β_e in (19) indicates that equities, because of the risk associated with them, are only a weak alternative to the almost risk-free government bonds. Hence, even if equities offer a markedly higher rate of return, the pressure of demand on the stock market and the price increases resulting from it are only limited. With respect to a given (positive) differential in the two rates of return, a higher value of β_e signifies a greater pressure of demand on the share price, since agents are more easily willing to switch from bonds to equities, or to invest their current savings in equities rather than bonds. The limit in which $\beta_e = \infty$ would correspond to the situation of perfect substitutability.

Eq. (20) invokes adaptive expectations for the expected capital gains, which are therefore treated analogously to the general inflation climate in (16). The speed of adjustment $\beta_{\pi e}$, however, may be typically faster than $\beta_{\pi p}$.

It is not very hard to set up versions of speculative stock market dynamics that specify some of the relevant features in finer detail. They may, in particular, explicitly distinguish fundamentalists and chartists as the two prototype trading groups on this market (see, e.g., Franke and Sethi, 1998), or the speeds of adjustments in eqs (19) or (20) could be dependent on the recent history of returns (*ibid.*) or the economic climate in general (as in Chiarella et al., 2002). The basic stabilizing and destabilizing feedback mechanisms, however, would be very similar, at least as as far as the local dynamics is concerned. For this reason the elementary adjustment equations (19) and (20) may for the present (but only for the present) suffice.

3 The model in intensive form

In order to analyze the dynamics generated by eqs (1) – (21), the model has to be translated into intensive form. In this way a seven-dimensional differential equation system comes about, with state variables:

y	$=$	Y/K	output-capital ratio
v	$=$	wL/pY	wage share
q	$=$	p_eE/pK	Tobin's q
π_e			expected capital gains
m	$=$	M/pK	real balances ratio
k	$=$	K/zL^s	capital per head
π			expected price inflation

Capital per head is a shorthand expression for capital per hour of labour supplied, measured in efficiency units (recall z is labour productivity).

The differential equations

We first sketch the way how the differential equations for the state variables are obtained, including the composite expressions of some of the variables entering them. For a better overview, the complete equations are subsequently collected in a whole.

The time rate of change of the output-capital ratio is obtained from $\dot{y} = \dot{Y} - \hat{K}y$ and, in particular, eq. (12). Making use of (7) – (11), the aggregate demand term $y^d = Y^d/K$ in (22) is easily computed as stated in eq. (29) below. In order to determine the rate of inflation in (16) and also the changes in the wage share, the mutual dependence of $\hat{p}$ and $\hat{w}$ in the two Phillips has to be eliminated. Defining $\kappa := 1/(1-\kappa_p\kappa_w)$, the reduced-form equations read

$$\begin{aligned}\hat{p} &= \pi + \kappa\,[\,\beta_p(y-y^n) + \kappa_p\beta_w(\lambda-1)\,]\\ \hat{w} &= \pi + g_z + \kappa\,[\,\kappa_w\beta_p(y-y^n) + \beta_w(\lambda-1)\,]\end{aligned}$$

On this basis, the changes in the wage share $v = wL/pY = w/pz$ derive from $\hat{v} = \hat{w} - \hat{p} - g_z$; see (23), with κ given in (34). Obviously, the two weights κ_p and κ_w must not both be unity simultaneously. Eq. (30) makes explicit how the employment rate entering (23) can be expressed in terms of the state variables y and k. This formula is the simple decomposition $\lambda = (Y/K)(L/Y)(K/L^s) = (Y/K)(K/zL^s)$. The equation for the rate of inflation will also be referred to in the next steps and so is reiterated in (31).

To get the law of motion for Tobin's q, observe first that, with (18), the growth rate of the number of equities is $\hat{E} = p_e\dot{E}/p_eE = (pI/pK)(pK/p_eE) = \hat{K}/q$. Thus, $\hat{q} = \hat{p}_e + \hat{E} - \hat{p} - \hat{K} = \hat{p}_e - \hat{p} + [(1-q)/q]\,\hat{K}$. Eq. (24) follows from substituting (19) for $\hat{p}_e$. As (32) shows, the rate of profit entering here is dependent on y and v. Furthermore, the bond rate i in (24) depends on y and m. It is the solution of the LM equation (17) in intensive form, $m = y\,(\beta_{mo} - \beta_{mi}\,i)$, given in eq. (33).

The adjustments of the expected capital gains in (25) combine eq. (20) with (19). The motions of the remaining three variables are less involved. Logarithmic differentiation of $m = M/pK$ and $k = K/zL^s$, together with the growth rate specifications (1) – (5), yields the differential equations (26) and (27) for these two variables. Lastly, (28) for the adaptive expectations of the inflation climate π is a restatement of (16).

$$\dot{y} = \beta_y\,[\,y^d - (y - g^o\beta_n y^n)\,] - (y - g^o\beta_n y^n)\,\beta_I\,(q-1) \tag{22}$$

$$\dot{v} = v\,\kappa\,[\,\beta_w(1-\kappa_p)(\lambda-1) - \beta_p(1-\kappa_w)(y-y^n)\,] \tag{23}$$

$$
\begin{aligned}
\dot q &= q\,\Big\{\beta_e\Big[\frac{r}{q}+\pi_e-\xi-i\Big] + \kappa_e\pi_e + (1-\kappa_e)\pi^o - \hat p \\
&\qquad + \; [(1-q)/q]\,[\,g^o+\beta_I(q-1)\,]\Big\} &(24)\\
\dot\pi_e &= \beta_{\pi e}\,\Big\{\beta_e\Big[\frac{r}{q}+\pi_e-\xi-i\Big] - (1-\kappa_e)\,(\pi_e-\pi^o)\Big\} &(25)\\
\dot m &= m\,[\pi^o - \hat p - \beta_I(q-1)] &(26)\\
\dot k &= k\,\beta_I\,(q-1) &(27)\\
\dot\pi &= \beta_{\pi p}\,(\hat p-\pi) &(28)
\end{aligned}
$$

$$
\begin{aligned}
y^d &= y^d(y,v,q) \;=\; (1-s_c)y + s_c\,vy + \beta_I(q-1) + g^o + s_c\delta+\gamma-(1-s_c)\theta &(29)\\
\lambda &= \lambda(y,k) \;=\; y\,k &(30)\\
\hat p &= \hat p(y,k,\pi) \;=\; \pi + \kappa\,[\,\beta_p(y-y^n)+\kappa_p\beta_w(\lambda-1)\,] &(31)\\
r &= r(y,v) \;=\; (1-v)y - \delta &(32)\\
i &= i(y,m) \;=\; \beta_{mo}/\beta_{mi} - m/(\beta_{mi}\,y) &(33)\\
\kappa &= 1\,/\,(1-\kappa_p\kappa_w) &(34)
\end{aligned}
$$

Long-run equilibrium

A steady state position of the economy is constituted by a rest point of system (22) – (28). Proposition 1 ensures that it exists and that, at least if the system is not degenerate, it is unique. The steady state values of the variables are denoted by a superscript ‘o’. Note that expected inflation π in the steady state is indeed equal to π^o as defined in (5), which justifies the slight slip in the notation for equilibrium inflation.

Proposition 1

System (22) – (28) has a stationary point given by

$$
\begin{aligned}
y^o &= y^n\,, & v^o &= 1 - [\,g^o(1+\beta_n y^n)+s_c\delta+\gamma-(1-s_c)\theta\,]\,/\,s_c y^n\\
q^o &= 1 & (\hat p)^o &= (\pi)^o \;=\; (\pi_e)^o \;=\; \pi^o\\
k^o &= 1/y^n & m^o &= [\,\beta_{mo} - \beta_{mi}((1-v^o)y^n-\delta+\pi^o-\xi)\,]\,y^n
\end{aligned}
$$

Provided that the parameters $\beta_I,\beta_p,\beta_w,\beta_{\pi p},\beta_e,\beta_{\pi e}$ are all strictly positive, this position is uniquely determined.

Proof: We proceed in a number of successive steps. Setting $\dot k=0$ gives $q=q^o=1$; setting $\dot m=0$ gives $\hat p^o=\pi^o$; setting $\dot\pi=0$ gives $(\pi)^o=\hat p^o=\pi^o$ for the inflation climate. Then, consider $\dot q=0$ and $\dot\pi_e=0$ (the terms in the curly brackets only). Subtracting the second equation from the first gives $[\kappa_e+(1-\kappa_e)](\pi_e-\pi^o)=0$, hence $(\pi_e)^o=\pi^o$.

From $\dot v=0$ we get $\beta_w(1-\kappa_p)(\lambda-1)-\beta_p(1-\kappa_w)(y-y^n)=0$, while from $\hat p=\pi$ in (31) we have $\beta_w\kappa_p(\lambda-1)+\beta_p(y-y^n)=0$. This can be viewed as two equations in the two unknowns

$(\lambda - 1)$ and $(y - y^n)$. An obvious solution is $(\lambda - 1) = 0$, $(y - y^n) = 0$, and it is easily seen to be the only one if κ_p and κ_w are not both unity simultaneously. $\lambda = 1$ implies $k = k^o = 1/y^n$ by eq. (30).

Putting $\dot{y} = 0$ and solving the resulting equation $y^n - g^o \beta_n y^n = (y^d)^o$ for v yields v^o as stated in the proposition. Returning to $\dot{q} = 0$ and setting $r^o = (1 - v^o) y^n - \delta$ in (32), this equation now amounts to $r^o + \pi^o - \xi - i = 0$, or $r^o + \pi^o - \xi = i(y^n, m) = \beta_{mo}/\beta_{mi} - m/(\beta_{mi} y^n)$ with (33). Solving this for m gives the proposition's expression for the equilibrium real balances.

q.e.d

4 Subdynamics in the real and financial sector

To get an impression of the basic stabilizing and destabilizing forces in the economy, it is instructive to study the dynamics of its underlying subsystems. One of them represents the stock market, the other the goods and labour market in the real sector.

Stock market subdynamics

Let us first consider the dynamics on the stock market in isolation of the rest of the economy. Two variables are here determined; Tobin's q reflecting the adjustments of equity prices, and expected capital gains π_e.

There is only one channel through which the stock market impacts on the real sector. This is Tobin‘s q in the investment function of the firms. Hence the real sector may continue to grow in its steady state proportions if the investment coefficient β_I in (8) is set equal to zero. Freezing the five state variables y, v, m, k, π (that characterize the behaviour of the real sector) at their equilibrium values and denoting $r^o = r(y^o, v^o)$, $i^o = i(y^o, m^o)$, the differential equations (24), (25) become

$$\dot{q} \;=\; q \left\{ \beta_e \left[\frac{r^o}{q} + \pi_e - \xi - i^o \right] \;+\; \kappa_e (\pi_e - \pi^o) \;+\; \frac{(1-q)\, g^o}{q} \right\} \tag{35}$$

$$\dot{\pi}_e \;=\; \beta_{\pi e} \left\{ \beta_e \left[\frac{r^o}{q} + \pi_e - \xi - i^o \right] \;-\; (1 - \kappa_e)\, (\pi_e - \pi^o) \right\} \tag{36}$$

An obvious destabilization mechanism is the positive feedback loop of expected capital gains. A rise in π_e increases the demand for equities and thus drives up share prices. If p_e rises sufficiently fast relative to general inflation, the gap between $\hat{p}_e$ and π_e in eq. (20) widens, so that π_e increases further. On the other hand, the resulting increase in Tobin's q lowers the rate of return on equities (the expression r^o/q), which tends to reduce equity demand. Whether the positive or negative feedback dominates depends on the speed at which expectations of capital gains are adjusted upward: instability (stability) should prevail if $\beta_{\pi e}$ is sufficiently high (low). Proposition 2 makes more precise the conditions under which this happens.

Proposition 2

> *Suppose the equilibrium rate of profit exceeds the real growth rate, $r^o > g$. Then the equilibrium q^o, π^o of system (35), (36) for the stock market subdynamics is locally asymptotically stable if either $\beta_e \le 1 - \kappa_e$, or (with $\beta_e > 1 - \kappa_e$) if*
>
> $$\beta_{\pi e} \;<\; (g^o + \beta_e r^o) \,/\, (\beta_e + \kappa_e - 1).$$
>
> *The equilibrium is unstable if in the latter case ($\beta_e > 1 - \kappa_e$) the inequality for $\beta_{\pi e}$ is reversed.*

The supposition of the relative size of the profit rate r^o can be safely taken for granted.[4] Furthermore, overly sluggish reactions of equity prices, as represented by low values of β_e, do not really seem plausible. The proposition's inequality for the adjustment speed of expected capital gains, $\beta_{\pi e}$, is therefore the central stability condition for the stock market.

Proof: The Jacobian of (35), (36) is given by

$$J \;=\; \begin{bmatrix} -(\beta_e r^o + g^o) & \beta_e + \kappa_e \\ -\beta_{\pi e}\beta_e \, r^o & \beta_{\pi e}(\beta_e + \kappa_e - 1) \end{bmatrix}$$

$r^o > g^o$ is sufficient for the determinant to be positive, since $\det J \;=\; \beta_{\pi e}\,[\,\beta_e(r^o - g^o) + g^o(1 - \kappa_e)\,]$. The statements in the proposition then derive from the second stability condition that the trace be negative.

q.e.d

The income distribution subdynamics

Neglecting variations in fixed investment by putting $\beta_I = 0$ not only decouples the real sector from the stock market, but there is also no feedback of money balances m on the goods market, since we do not have to consider the impact of the bond rate $i = i(y, m)$ on Tobin's q. Furthermore, capital per head k remains constant in the employment rate $\lambda = \lambda(y, k)$. For simplicity, let us here also ignore the inflationary climate and its influence on current inflation $\hat{p}$ in eq. (31) by fixing π at π^o. In this way we concentrate on a two-dimensional subdynamics in output y and the wage

[4] Indeed, taxes must be extraordinarily (unless inconsistently) high for the inequality $r^o > g^o$ to be violated. This can be seen from substituting the expression for v^o from Proposition 1 in $r(y^o, v^o)$, which yields $r^o = [\,g^o(1 + \beta_n y^n) + \gamma - (1 - s_c)(\delta + \theta)\,]$.

share v, which reads,[5]

$$\dot{y} \;=\; \beta_y\,[\,y^d(y,v,q^o) - y + g^o\beta_n y^n\,] \tag{37}$$

$$\dot{v} \;=\; v\,\kappa\,\{\,\beta_w(1-\kappa_p)[\lambda(y,k^o)-1] \;-\; \beta_p(1-\kappa_w)(y-y^n)\,\} \tag{38}$$

The stability of this reduced system crucially depends on the relative speeds at which wages and prices respond to the disequilibrium on the labour and goods market. To see this, suppose a positive shock on the wage side has raised the wage share above its steady state level. The immediate effect is an increase in consumption demand on the part of workers. Since there are no possibly counterbalancing effects in investment, aggregate demand y^d and then total output y increases. The corresponding overutilization of the capital stock raises inflation in the price Phillips curve (14), while the correspondingly higher employment rate λ raises wage inflation in (15). If the price level rises faster than the nominal wage rate (discounting for the productivity growth rate g_z in the wage Phillips curve), the wage share falls back towards normal. In this way we identify a negative, stabilizing feedback loop. Otherwise the wage share increases and moves the economy further away from equilibrium.

This short chain of effects may be called the real wage effect, or the Rose effect, whereby we pay tribute to a seminal contribution on the stability implications of wage and price adjustments by Rose (1967) or, more comprehensively later on, by Rose (1990). Normally this effect may be expected to be stabilizing, so we may speak of an adverse Rose effect in the destabilizing case. Schematically, the two cases may be summarized as follows, where $\uparrow\uparrow$ indicates a faster rate of change than $\uparrow$:

$$\text{Normal Rose effect:} \qquad v\uparrow \;\longrightarrow\; y^d\uparrow \;\longrightarrow\; \left\{ \begin{array}{lll} y\uparrow & \longrightarrow & \hat{p}\uparrow\uparrow \\ \lambda\uparrow & \longrightarrow & \hat{w}\uparrow \end{array} \right. \;\longrightarrow\; v\downarrow$$

$$\text{Adverse Rose effect:} \qquad v\uparrow \;\longrightarrow\; y^d\uparrow \;\longrightarrow\; \left\{ \begin{array}{lll} y\uparrow & \longrightarrow & \hat{p}\uparrow \\ \lambda\uparrow & \longrightarrow & \hat{w}\uparrow\uparrow \end{array} \right. \;\longrightarrow\; v\uparrow$$

Proposition 3 shows that in the real sector subsystem the sign of the Rose effect is the decisive stability argument. The key expression for stability is α_{wp}, which contrasts the adjustment speeds β_w and β_p in the two Phillips curves,

$$\alpha_{wp} \;=\; \beta_w(1-\kappa_p)/y^n \;-\; \beta_p(1-\kappa_w) \tag{39}$$

[5] Adopting the (otherwise useful) notation $y^d = y^d(\cdot,\cdot,\cdot)$ for aggregate demand, the supposition $\beta_I = 0$ is expressed by plugging in q^o. This nevertheless does not rule out that q may be actually moving on the stock market.

Note, however, that α_{wp} not only involves β_w and β_p, but also the weighting parameters κ_w and κ_p in (14), (15).

Proposition 3

The equilibrium point y^o, v^o of the real sector subsystem (37), (38) is locally asymptotically stable if $\alpha_{wp} < 0$. It is unstable if the inequality is reversed.

Proof: The Jacobian is

$$J \;=\; \begin{bmatrix} -s_c\beta_y(1-v^o) & s_c\beta_y\,y^o \\ v^o\kappa\,\alpha_{wp} & 0 \end{bmatrix}$$

Its trace is always negative, and $\det J = -s_c\beta_y\,y^o v^o\,\kappa\,\alpha_{wp}$. Hence stability is given if and only if $\det J > 0$, that is, if and only if α_{wp} is negative. *q.e.d*

In evaluating the Rose effect, the proposition warns against exclusively looking at the direct wage and price adjustment speeds. Though conceptually the weights κ_w and κ_p are of secondary importance, their dynamic implications for the real wage, or the wage share, are by no means innocent.[6] In particular, if price adjustments are independent of the inflation climate, i.e. if $\kappa_p = 1$, the Rose effect will always be normal, even if β_w itself might be excessively high.[7] A converse reasoning applies for $\kappa_w = 1$. The distorting effect of κ_w and κ_p is more directly expressed if (assuming $\kappa_p < 1$) the stability condition is rewritten as

$$\beta_w \;<\; \frac{(1-\kappa_w)\,y^n}{1-\kappa_p}\,\beta_p \tag{40}$$

It may finally be remarked that though it is tempting to relax the assumption $\beta_I = 0$ and merge the stock market dynamics (35), (36) with the real sector subdynamics (37), (38), the resulting four-dimensional system would not be consistent, even if π were still kept at π^o. On the one hand, k is now being changed in (27) by the variations of Tobin's q, which feeds back in $\lambda = \lambda(y,k)$ in (38). On the other hand, the variable real balances m from (26) make themselves felt in the interest rate $i = i(y,m)$ in (35). In addition, such a combined system would not be easily tractable, either. Results going beyond what can also be obtained for the general system would be hard to come by. We therefore proceed directly to the stability analysis of the full 7th-order dynamics that integrates consistently the stock market and real sector dynamics.

[6] Empirical estimations of Phillips curves, however, do not seem to pay much attention to coefficients like κ_w and κ_p. They are possibly fairly sensitive to the specific proxy adopted for inflationary expectations.

[7] We recall that, for κ in (34) to be well-defined, κ_w cannot be unity, too, in this case.

5 Local stability analysis of the full 7D dynamics

Immediate instability results

The local stability analysis of the full seven-dimensional differential equations system (22) – (28) is based on the Jacobian matrix J, evaluated at the steady state values. It is useful in this respect to change the order of the differential equations. Maintaining the first three for y, v, q and rearranging the remaining four in the order m, k, π, π_e, the Jacobian is computed as

$$J = \begin{bmatrix} -s_c\beta_y(1-v) & s_c\beta_y y & \beta_I\,\alpha_{yn} & 0 & 0 & 0 & 0 \\ v\kappa\,\alpha_{wp} & 0 & 0 & 0 & v\kappa\,\alpha_{w\kappa} & 0 & 0 \\ \beta_e\,\alpha_{vi} - \hat{p}_y & -\beta_e y & -(\beta_e r + g) & -\beta_e\, i_m & -\hat{p}_k & -1 & \beta_e + \kappa_e \\ -m\,\hat{p}_y & 0 & m\,\beta_I & 0 & -m\,\hat{p}_k & -m & 0 \\ 0 & 0 & k\,\beta_I & 0 & 0 & 0 & 0 \\ \beta_{\pi p}\,\hat{p}_y & 0 & 0 & 0 & \beta_{\pi p}\,\hat{p}_k & 0 & 0 \\ \beta_{\pi e}\beta_e\,\alpha_{vi} & -\beta_{\pi e}\beta_e\, y & -\beta_{\pi e}\beta_e\, r & -\beta_{\pi e}\beta_e\, i_m & 0 & 0 & \beta_{\pi e}(\beta_e+\kappa_e-1) \end{bmatrix}$$

Here the superscript 'o' is omitted and besides α_{wp}, which has already be defined in (39), the following abbreviations are used:

$$\begin{aligned} \alpha_{yn} &= \beta_y - (1-g\beta_n)\,y & \qquad \hat{p}_y &= \partial\hat{p}/\partial y = \kappa\,[\beta_p + \beta_w\kappa_p k] \\ \alpha_{vi} &= (1-v) - \partial i/\partial y & \hat{p}_k &= \partial\hat{p}/\partial k = \kappa\,\beta_w\kappa_p y \\ \alpha_{w\kappa} &= \beta_w\,(1-\kappa_p)\,y & i_m &= \partial i/\partial m = -1\,/\,\beta_{mi}\,y \end{aligned}$$

One of the necessary conditions for local stability is a negative trace. With $\beta_e > 1-\kappa_e$, this condition is obviously violated whenever $\beta_{\pi e}$ in the diagonal entry j_{77} of matrix J is large enough. We can thus immediately recognize that the stock market dynamics can always destabilize the whole economy: if the expected capital gains adjust sufficiently fast to the previously observed changes in the equity price.

Proposition 4

> *Given that $\beta_e > 1-\kappa_e$, the steady state of system (22) – (28) is unstable if $\beta_{\pi e}$ is sufficiently large.*

In many monetary growth models with adaptive expectations of an expected rate of inflation, the same type of result obtains if the speed of adjustment of inflationary expectations is sufficiently fast. In the present framework this would mean that $\beta_{\pi p}$, too, could destabilize the economy. However, $\beta_{\pi p}$ does not show up in the diagonal entry j_{66}, so that the straightforward argument involving the trace of J is no longer available. As a matter of fact, it can be shown below that low values of $\beta_{\pi p}$ are stabilizing, whereas we have so far not been able to prove that high values of $\beta_{\pi p}$ are (largely) sufficient for instability.

Another mechanism that, at least at a theoretical level, is likewise capable of destabilizing the whole economy is an adverse Rose effect. The mathematical argument refers to the principal minor of order 2 which is constituted by the determinant of the 2×2 submatrix in the upper-left corner. Denoting it by $D^{(2)}$, we have, as in the proof of Proposition 3, $D^{(2)} = -s_c \beta_y \, y^o v^o \, \kappa \, \alpha_{wp}$. One of the more involved Routh-Hurwitz conditions necessary for stability says that the sum of all second-order principal minors must be positive. Since α_{wp} enters no other of these principal minors, a negative $D^{(2)}$ can dominate the sum if α_{wp} is large enough. We thus obtain

Proposition 5

> *Given that $\kappa_p < 1$, the steady state of system (22) – (28) is unstable if β_w is sufficiently large.*

After these negative results we should now inquire into the conditions for the long-run equilibrium growth path to be attractiving.

The proof strategy: A cascade of stable matrices

Stability conditions for system (22) – (28) can be derived in a number of successive steps, where we proceed from lower- to higher-order matrices. Our method rests on the following lemma.

Lemma

> *Let $J^{(n)}(\beta)$ be $n \times n$ matrices, $h(\beta) \in \mathbb{R}^n$ row vectors, and $h_{n+1}(\beta)$ real numbers, all three varying continuously with β over some interval $[0, \varepsilon]$. Put*
>
> $$J^{(n+1)}(\beta) \;=\; \begin{bmatrix} J^{(n)}(\beta) & z \\ h(\beta) & h_{n+1}(\beta) \end{bmatrix} \;\in\; \mathbb{R}^{(n+1)\times(n+1)} \,,$$
>
> *where z is an arbitrary column vector, $z \in \mathbb{R}^n$. Assume $h(0) = 0$, $\det J^{(n)}(0) \neq 0$, and let $\lambda_1, \ldots, \lambda_n$ be the eigen-values of $J^{(n)}(0)$. Furthermore for $0 < \beta \leq \varepsilon$, $\det J^{(n+1)}(\beta) \neq 0$ and of opposite sign of $\det J^{(n)}(\beta)$. Then for all positive β sufficiently small, n eigen-values of $J^{(n+1)}(\beta)$ are close to $\lambda_1, \ldots, \lambda_n$, while the $n+1$st eigen-value is a negative real number. In particular, if matrix $J^{(n)}(0)$ is asymptotically stable, so are these matrices $J^{(n+1)}(\beta)$.*

Proof: With respect to $\beta = 0$, it is easily seen that $J^{(n+1)}(0)$ has eigen-values $\lambda_1, \dots, \lambda_n, h_{n+1}(0)$. In fact, if λ is an eigen-value of $J^{(n)}$ with right-hand eigen-vector $x \in \mathbb{R}^n$, the column vector $(x, 0) \in \mathbb{R}^{n+1}$ satisfies $J^{(n+1)}(x, 0) = \lambda \cdot (x, 0)$. (We omit reference to the argument β for the moment.) This shows that λ is an eigen-value of $J^{(n+1)}$, too. It is furthermore well-known that the product of the eigen-values of a matrix equals its determinant, which gives us $\det J^{(n)} = \lambda_1 \cdot \ldots \cdot \lambda_n$ and $\det J^{(n+1)} = \lambda_1 \cdot \ldots \cdot \lambda_n \cdot \lambda_{n+1}$. On the other hand, expanding $\det J^{(n+1)}$ by the last row yields $\det J^{(n+1)} = h_{n+1}(0) \cdot \det J^{(n)} \neq 0$. Hence $\lambda_{n+1} = h_{n+1}(0)$.

Then, consider the situation in the lemma and denote the $n+1$ eigen-values of $J^{(n+1)}(\beta)$ by $\lambda_i(\beta)$. It has just been shown that $\lambda_i(0) = \lambda_i$ for $i = 1, \dots, n$, while $\lambda_{n+1}(0)$ is a real number. Eigen-values vary continuously with the entries of the matrix.[8] As $\det J^{(n)}(0) \neq 0$, this implies that $\text{sign}[\lambda_1(\beta) \cdot \ldots \cdot \lambda_n(\beta)] = \text{sign}[\lambda_1 \cdot \ldots \cdot \lambda_n] = \text{sign}[\det J^{(n)}(0)]$ also for small positive β. The relationship $\det J^{(n+1)}(\beta) = \lambda_1(\beta) \cdot \ldots \cdot \lambda_n(\beta) \cdot \lambda_{n+1}(\beta)$ entails $\text{sign}[\det J^{(n+1)}(\beta)] = \text{sign}[\det J^{(n)}(0)] \cdot \text{sign}[\lambda_{n+1}(\beta)] \neq 0$. Since $\det J^{(n+1)}(\beta)$ and $\det J^{(n)}(0)$ have opposite signs, $\lambda_{n+1}(\beta)$ is a negative real number for all β sufficiently small (but, of course, β still positive, should $h_{n+1}(0)$ happen to be zero).

The final statement about the stability of $\det J^{(n+1)}(\beta)$ follows from the fact that, by hypothesis, the n eigen-values of $\det J^{(n)}(0)$ have all strictly negative real parts. So for small β the same holds true for the $\lambda_i(\beta)$.

q.e.d

We thus proceed with the above Jacobian matrix in the following manner. Suppose in the n-th step, so to speak, a submatrix $J^{(n)}$ made up of the first n rows and columns of J, $n < 7$, has been established to be stable. Suppose moreover that there exists a parameter β such that all entries of the $n+1$st row, except perhaps for the diagonal entry $j_{n+1,n+1}$, converge to zero as $\beta \to 0$. If we are able to verify that the determinant of the augmented matrix $J^{(n+1)} = J^{(n+1)}(\beta)$ has the opposite sign of $\det J^{(n)}(\beta)$, the lemma applies and we conclude that $J^{(n+1)}(\beta)$ is stable as well if only β is chosen sufficiently small.

In this way a collection of parameter values are found that render the submatrix consisting of the first $n+1$ rows and columns of J stable. This result completes the $n+1$st step and we can go over to consider matrix $J^{(n+2)}$, etc. On the whole, we therefore strive to obtain a cascade of stable matrices $J^{(n)}$, $J^{(n+1)}$, $J^{(n+2)}$, ... , until at $n = 7$ stability of the full system has been proved.

The argument, of course, equally applies if it is the $n+1$st column that exhibits the property

[8] This proposition is so intuitive that it is usually taken for granted. Somewhat surprisingly, a rigorous proof, which indeed is non-trivial, is not so easy to find in the literature. One reference is Sontag (1990, pp. 328ff).

just indicated. Likewise, if β does not enter matrix $J^{(n)}$, the $n+1$st column or row may also converge to zero as β itself tends to infinity.

Local stability of the full system

We are now ready to consider the full dynamical system (22) – (28) and put forward conditions on the behavioural parameters that ensure local stability of its long-run equilibrium.

Proposition 6

Consider the non-degenerate system (22) – (28) (in particular, $\beta_p > 0$, $\beta_I > 0$). Suppose that $\kappa_w < 1$, $\beta_I < s_c r^o$, and $\beta_y > (1 - g^o\beta_n)\, y^o$. Then the steady state position is locally asymptotically stable if β_w, $\beta_{\pi p}$, $\beta_{\pi e}$ are sufficiently small, while β_{mi} is sufficiently large.

Proof: Our starting point is the submatrix $J^{(3)}$ given by the first three rows and columns of J. For the moment being, we assume that $\beta_w = 0$ (so that $\alpha_{wp} = -\beta_p(1-\kappa_w) < 0$) and $\beta_{mi} = \infty$ (so that $\partial i/\partial y = 0$ and $\alpha_{vi} = 1 - v$). Given that $\beta_I < s_c r^o$, it has to be shown that $J^{(3)}$ satisfies the Routh-Hurwitz conditions. This demonstration may be summarized as 'steps 1–3'.

<u>*Steps 1–3:*</u> The Routh-Hurwitz conditions require that the following terms a_1, a_2, a_3, b are all positive. Again, here and in the rest of the proof superscripts 'o' are omitted.

$$
\begin{aligned}
a_1 &= -\operatorname{trace} J &&= |j_{11}| + |j_{33}| \;=\; s_c\beta_y(1-v) + (\beta_e r + g)\\
a_2 &= J_1^{(3)} + J_2^{(3)} + J_3^{(3)} &&= 0 + [\,\beta_y\beta_e(1-v)(s_c r-\beta_I) + \kappa\beta_p\alpha_{yn} + a_{21}\,] + v\kappa y\, s_c\beta_y\, |\alpha_{wp}|\\
a_3 &= -\det J^{(3)} &&= |\alpha_{wp}|\, v\kappa y\, [\,\beta_y\beta_e(s_c r-\beta_I) + s_c\beta_y + \beta_e\beta_I(1-g\beta_n)y\,]\\
& &&= |\alpha_{wp}|\, v\kappa y\, \beta_y s_c\,(\beta_e r + g) \;-\; |\alpha_{wp}|\, v\kappa y\, \beta_e\beta_I\, \alpha_{yn}\\
& &&=: a_{31} - a_{32}\\
b &= a_1\, a_2 - a_3 &&= (|j_{11}| + |j_{33}|)\,(J_2^{(3)} + J_3^{(3)}) - a_{31} + a_{32}
\end{aligned}
$$

Positivity of a_1 is obvious. a_{21} in the determination of $J_2^{(3)}$ is a positive residual term, while $\alpha_{yn} > 0$ by the assumption on β_y. Hence $a_2 > 0$ by virtue of $s_c r-\beta_I > 0$. The latter inequality also ensures $a_3 > 0$, as seen in the first line of the computation of a_3. Decomposing a_3 as indicated in the second line allows us to infer $b > 0$. It suffices to note here that $-a_{31}$ cancels against $|j_{33}| \cdot J_3^{(3)}$, so that only positive terms remain.

<u>*Step 4:*</u> Regarding $J^{(4)}$, consider its 4th column and take $i_m = \partial i/\partial m = -1/\beta_{mi}\, y$ as the relevant parameter when referring to the lemma. Since $i_m \to 0$ as $\beta_{mi} \to \infty$, and $\det J^{(3)} < 0$, we have to show that $\det J^{(4)} > 0$ for $i_m \neq 0$ (i.e., $\beta_{mi} < \infty$). In fact, expanding $\det J^{(4)}$

by the 4th column and the remaining determinant by the 2nd column, it is easily seen that $\det J^{(4)} = \beta_e i_m\,(-s_c\beta_I)\,v\kappa\,\alpha_{wp}\,(-m\beta_I) > 0$ (recall $i_m < 0$, $\alpha_{wp} < 0$).

<u>*Step 5:*</u> Realize that in the 5th column of $J^{(5)}$, α_{wk} and $\hat{p}_k$ tend to zero as $\beta_w \to 0$. It therefore remains to verify that the determinant of $J^{(5)}$ is negative as $\beta_w > 0$. This can be seen by expanding $\det J^{(5)}$ by the 4th column, the newly arising determinant by the 4th row, and the next one by the 2nd column, which yields $\det J^{(5)} = \beta_e i_m\,(-k\beta_I)\,(-s_c\beta_y y)\,v\kappa m \cdot \det \tilde{J}^{(2)}$, where

$$\det \tilde{J}^{(2)} \;=\; \det \begin{bmatrix} \alpha_{wp} & \alpha_{wk} \\ -\hat{p}_y & -\hat{p}_k \end{bmatrix} \;=\; y\kappa\,\beta_p\,\beta_w\,(1-\kappa_p\kappa_w) \;>\; 0$$

<u>*Step 6:*</u> For $J^{(6)}$, the lemma applies with respect to the 6th row and $\beta_{\pi p} \to 0$, so only $\det J^{(6)} > 0$ has to be shown. While the previous determinants could be computed directly, it is here useful to carry out certain row operations that leave the value of the determinant unaffected but lead to a convenient structure of zero entries in the matrix. Apart from that, a couple of multiplicative terms are factorized (they do not involve a sign change since they are all positive). To ease the presentation, we do all this directly for the final matrix $J^{(7)} = J$, the result being the determinant $D^{(7)}$ below. Observe, however, that none of the first six rows is modified by adding or subtracting the 7th row. Hence the determinant of the first six rows and columns in $D^{(7)}$ exhibits the same sign as $\det J^{(6)}$. In detail, $D^{(7)}$ is obtained as follows.

1. Factorize $v\kappa$, m, k, $\beta_{\pi p}$, $\beta_{\pi e}$ in row 2, 4, 5, 6, 7, respectively.
2. Add the 5th row to the 4th row, to let entry j_{43} vanish.
3. Add the (new) 4th row to the 6th row, which makes entries j_{61} and j_{65} vanish, while j_{66} becomes -1.
4. Subtract the (new) 4th row from the 3rd row, so that $\hat{p}_y$ disappears from j_{31}, and j_{35}, j_{36} both become zero.
5. Subtract the (new) 3rd row from the 7th row, in which way $j_{71} = j_{72} = j_{74} = j_{75} = j_{76} = 0$, $j_{77} = -1$, and $j_{73} = g^o$.
6. Subtract g^o/β_I times the 5th row from the 7th row, which renders j_{73} zero and leaves the other entries in that row unaffected.

$$D^{(7)} = \det \begin{bmatrix} -s_c\beta_y(1-v) & s_c\beta_y y & \beta_I\,\alpha_{yn} & 0 & 0 & 0 & 0 \\ \alpha_{wp} & 0 & 0 & 0 & \alpha_{w\kappa} & 0 & 0 \\ \beta_e\,\alpha_{vi} & -\beta_e y & -(\beta_e r^o + g^o) & -\beta_e\,i_m & 0 & 0 & \beta_e + \kappa_e \\ -\hat{p}_y & 0 & 0 & 0 & -\hat{p}_k & -1 & 0 \\ 0 & 0 & \beta_I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Regarding $D^{(6)}$, the determinant of the first six rows and columns in $D^{(7)}$, it is easily checked that

$$\begin{aligned} D^{(6)} &= (-1)\,(\beta_e i_m)\,(-s_c\beta_y y)\,(-\beta_I)\,v\kappa m \cdot \det \tilde{J}^{(2)} \\ &= \beta_e\,|i_m|\,s_c\beta_y y\,\beta_I\;v\kappa m\,y\kappa\,\beta_p\,\beta_w\,(1-\kappa_p\kappa_w) \end{aligned}$$

and therefore $\det J^{(6)} > 0$.

<u>*Step 7:*</u> As for $J^{(7)} = J$, the lemma applies with respect to the 7th row and $\beta_{\pi e} \to 0$. Since $D^{(7)} = (-1)\cdot D^{(6)}$, $\det J^{(7)}$ if of opposite sign of $\det J^{(6)}$. Summarizing that β_{mi} is sufficiently large in step 4 and β_w, $\beta_{\pi p}$, $\beta_{\pi e}$ are sufficiently small in steps 5, 6, 7, respectively, this completes the proof.

q.e.d

The condition $\beta_I < s_c\,r$ in Proposition 6 is reminiscent of similar formulations in many Keynesian-oriented macro models with explicit reference to a rate of profit and a propensity to save out of profit (or rental) income. There, for example, such an inequality ensures a negative slope of an IS-curve. The analogy is somewhat surprising since not only is the present model more complex, but also the investment reaction coefficient β_I refers to a variable, Tobin's q, which is usually not considered in these lower-order macro models. On the other hand, a closer look at the Routh-Hurwitz terms a_2 and a_3 in the proposition's proof shows that the condition $\beta_I < s_c\,r$ is not a necessary one and, depending on the relative size of various other parameters, there is some room for relaxing it.

The main result of the stability analysis can be succinctly summarized by saying that slow reactions or adjustments of the following kind are favourable for stability: sluggish reactions of the bond rate of interest, as brought about by a high interest elasticity of money demand (captured by the parameter β_{mi}); sticky adjustments of nominal wages (low parameter β_w); slow revisions of the inflationary climate variable (low adjustment speed $\beta_{\pi p}$); and slow adjustments of expected capital gains (low adjustment speed $\beta_{\pi e}$).

Cyclical dynamics

As concerns the parameters whose stabilizing effects have just been discussed, it may be conjectured that values of them lying in the other extreme are conducive to instability. Propositions 4 and 5 have confirmed this with respect to high values of β_w and $\beta_{\pi e}$, but it has been indicated that similar statements for other coefficients would be much harder to obtain. We nevertheless know what happens if the system loses its stability. As a side result of the stability proof it is easily inferred that the transition from the stable to the unstable case occurs by way of a Hopf bifurcation.

Proposition 7

> *Let the steady state of (the non-degenerate) system (22) – (28) be locally asymptotically stable. Consider a parameter, generically denoted by α, and suppose that under continuous* ceteris paribus *changes of α the steady state becomes unstable at some critical value α_H. Then at α_H the system undergoes a Hopf bifurcation.*

Proof: Step 7 in the proof of Proposition 6 has established that the Jacobian J is non-singular for all admissible (non-zero) values of the parameters. This implies that if the eigen-value $\lambda = \lambda(\alpha)$ with largest real part crosses the imaginary axis in the complex plane, at $\alpha = \alpha_H$, we have a pair of purely imaginary eigen-values, $\lambda(\alpha_H) = \pm i\, b$ in the usual notation. This is the key condition for a Hopf bifurcation to occur. The (very) technical details connected with the Hopf theorem are largely avoided (in particular, the velocity condition) if one uses the version of theorem A presented in Alexander and Yorke (1978, pp. 263–266).

q.e.d

A Hopf bifurcation asserts that for some interval of parameter values close to α_H, strictly periodic orbits of the dynamical system exist (which may be attracting or repelling). While we do not wish to overstate this phenomenon as such, we emphasize the more general feature associated with this result, namely, that the dynamics is determined by complex eigen-values. We can therefore conclude that there is broad scope for the economy to exhibit cyclical behaviour, which at the present state of the investigation may be dampened or undampened.

6 Conclusion

The model put forward in this paper has combined a financial sector, with special emphasis on the stock market, and a real sector that allows for disequilibrium on the goods as well as labour market. The model is still Keynesian-oriented but gives a greater role to income distribution, as it is determined by Phillips curves adjustments for both price and wage inflation, and it drops the usual assumption that bonds and equities are perfect substitutes. Although the feedbacks considered are already so rich that in the end a 7th-order differential equations system is obtained, it was possible to derive economically meaningful conditions for the local stability of the long-run equilibrium position. In particular, we pointed out the significance of normal Rose (real wage) effects and sufficiently slow adjustments of the expected capital gains. On the whole, the model provides a consistent framework for subsequent theoretical and empirical studies of the real-financial interaction.

Regarding future work on the model we should recall that goods market disequilibrium as we have perceived it implies the presence of inventories as buffers for excess demand or supply. It has been assumed that deviations of inventories from some target level do not feed back on the production decisions of firms. This restriction is legitimate to simplify the model, but it should be improved upon if it is desired to study the medium and long-run evolution of the economy, especially if we are to turn to the global dynamics. A proven model building block in this respect is a Metzlerian inventory mechanism along the lines of Franke (1996), Chiarella and Flaschel (2000, Ch. 6.3), Chiarella et al. (2000, Ch. 2.3.2).

Another topic of future research may concern the market for equities. It may be noted that a generalization of the standard Keynesian LM-sector can go in two directions. One option is to use the Tobinian portfolio approach that formulates the demand for the additional assets (besides money and bonds) as stock magnitudes. Inspired by Blanchard (1981), we followed in this paper the second direction where equities are treated somewhat differently from money and bonds, such that specifically the variable that represents the stock market, Tobin's q, becomes a dynamic variable.[9] The advantage of this approach is that the structure of the model would not be essentially affected if the present equity price adjustments are extended to a more clearly conceived speculative asset price dynamics. Thus, as indicated at the end of Section 2, the two prototype trading groups of fundamentalists and chartists could be explicitly discussed. Moreover, considering their trading strategies in finer detail we could introduce suitable and economically well-motivated nonlinearities that prevent the stock market dynamics from diverging.

If in line with Propositions 2 and 4 a high responsiveness of expectations about capital gains is viewed as the main destabilizing force for the whole model, then features that contain the stock market dynamics become particularly important. An additional point is that already very elementary mechanisms are capable of generating complex ('chaotic') dynamics.[10] It would

[9] In a portfolio approach, q would be determined as a statically endogenous variable, as part of the temporary equilibrium solution of the financial sector; see Franke and Semmler (1999).

[10] Various approaches to tackle the issue of boundedness can be found in Sethi (1996), Franke and Sethi(1998),

be interesting to see how this kind of speculative dynamics interacts with the ('normal') rest of the model. Clearly, these themes call for a global analysis that we leae for future research.

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