Dynamic Analysis of Policy Lag in a Keynes-Goodwin Model : Stability, Instability, Cycles and Chaos

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Abstract

In this paper, we investigate the impact of government's stabilization policy on the dynamic behavior of the economic system in an analytical framework of a Keynes-Goodwin model of the growth cycle. In particular, we study the effects of the policy lag on macroeconomic stability analytically and numerically. It is shown that the increase of the policy lag contributes to destabilize the system, and cyclical behavior and chaotic motion emerge in some ranges of the parameter values.

Key words : Keynes-Goodwin model ; Growth cycle ; Policy lag ;
Hopf bifurcation ; Chaotic motion ; Numerical simulations

JEL classification : E12, E30, E62

1. Introduction

The discussion on stabilization policies has been one of the most important topics in the macroeconomic literature, ever since Keynes(1936) established the notion of effective demand in his classical book, *The General Theory*. His main argument is that our *laissez-faire* capitalist economy tends to settle down to a position of underemployment without government intervention ; the government should have active control over the level of aggregate demand to achieve full employment. However, there are several opposite views to the effectiveness of Keynesian demand management policy. For example, Friedman(1948) asserted that the existence of policy lag could be a destabilizing factor. Although Friedman(1948)'s argument is not based on the full-fledged dynamic model of macroeconomic interdependency, Phillips(1954, 1957) studied the effects of policy lag on macroeconomic stability by means of the numerical analysis, which is explicitly based on a simple macrodynamic model of the multiplier-accelerator type.

After Phillips(1954, 1957), there exist only a few formal models of policy lag which are based on the explicit consideration of macrodynamic system of interdependency. Takamasu(1995) and Asada and Yoshida(2001) are two examples of such works. Takamasu(1995) introduced the policy lag into Goodwin(1967)'s growth cycle model, and showed the possibility of the chaotic movement by means of the numerical simulations. It can be considered that Takamasu(1995) is an extension of Wolfstetter(1982)'s pioneering work which discussed the economic implications of government's stabilization policy by introducing the fical policy *without policy lag* into Goodwin's growth cycle model. However, Goodwin(1967)'s original formulation, on which Takamasu(1995) heavily depends, has some difficulties as a description of the dynamics of the modern capitalist economy. Goodwin(1967)'s model consists of two main elements. First element is dynamic of income distribution which is based on the real wage Phillips curve, and second element is dynamic of employment which is governed by the capital accumulation. But, in this model, there is no room for Keynesian effective demand to play the active role, because Goodwin(1967) closed the model by assuming a sort of 'Say's law'. In other words, in his model, it is assumed that firms' investment expenditures are automatically adjusted to the levels of capitalists' savings to ensure the full capacity utilization of the existing capital stock, and there is no room for the investment function which is independent of capitalists' saving function. On the other hand, Asada and Yoshida(2001) introduced the policy lag into Kaldorian business cycle model which is originated in

Kaldsor(1940), and investigated the dynamics of the system analytically and numerically. In the analytical framework which was adopted by Asada and Yoshida(2001), the investment function which is independent of the saving function plays an active role, and the output fluctuates according to the Keynesian principle of effective demand instead of the Say's law. Contrary to Goodwin's approach, however, in this Kaldorian framework, the dynamics of wages and prices are absent, and economic growth is also abstracted from.

In this paper, we reconsider the macroeconomic implications of the policy lag by utilizing the analytical framework of the so called 'Keynes-Goodwin model', which was developed by Asada(1989), Skott(1989), Franke and Asada(1994) and others. This model also considers the dynamics of income distribution and capital accumulation, but, contrary to the original Goodwin model which consists of only real dynamics, in this model the nominal wage dynamics and price dynamics are considered separately. Furthermore, in our analytical framework, the active role of the independent investment function and the variable capacity utilization of the capital stock which is determined by the Keynesian / Kaleckian principle of effective demand are explicitly considered. (In these respects, our 'Keynes-Goodwin' model of growth cycle is similar to the 'Keynes-Wicksell' model which is due to Chiarella and Flaschel(1996) and 'Keynes-Goodwin-Metzler' model (KGM model) which is due to Chiarella, Flaschel, Groh and Semmler(2000), but structure of our model is simpler than their models.)

By the way, we add a rather technical matter. This paper considers time lags of fiscal policies by employing a continuously distributed lag, although we also consider the case of the fixed time lag as a limiting case. There exist several works which use the distributed lag within Goodwin's framework of the growth cycle. They share the same standpoint ; they modify the usual Phillips curve to allow for the time delay in the labor market. Chiarella(1990, Chap. 5) and Farkas and Kotsis(1992), for instance, emphasize that the wage rate adjusts sluggishly to the labor market disequilibrium. Another example is Fanti and Manfredi(1998). They pay attentions to a profit-sharing rule and the existence of asymmetric information in the wage determination. In contrast, our concern is to examine the effects of policy lag on governments' stabilization policy.

This paper is organized as follows. Based on the basic model which is formulated in section 2, section 3 studies the effects of the governments' stabilization policy by means of fiscal policy *without policy lag*. Section 4 extends the basic model to consider the policy lag and investigate the natures of the solutions analytically. Our model is formulated as a system of integro-differential equations, but, fortunately, it is shown that we can reduce it to a set of ordinary differential equations by using the so called 'linear chain trick'. It is also shown that a limiting case of this system becomes a set of differential-difference equations (delay-differential equations). In section 4, we investigate the local stability / instability properties of the system analytically and prove the existence of the limit cycle by using Hopf bifurcation theorem in a particular case of more general model. Section 5 is devoted to the numerical simulations of the limiting case of the fixed time lag, which is described by a set of differential-difference equations, and we show the possibility of the chaotic movement which is due to the policy lag numerically. Concluding remarks are given in section 6. In the appendix, some purely mathematical results, which are not only useful for our purposes but also have potential applicability to wider classes of macrodynamic analyses, are stated and proved.

2. Formulation of the basic model

The symbols used throughout this paper are defined as follows.[1]

Y = gross real output (gross real national income), K = real capital stock, N = labor employment, C_W = workers' real consumption expenditure, C_K = capitalists' real consumption expenditure, I = gross real private investment expenditure, G = real government expenditure, T = real income tax, B = nominal stock of government bond, p = price level, w = nominal wage rate, p^e = expected rate of price inflation, i = nominal rate of interest of government bond, p_B = market price of government bond, d = rate of depreciation of capital stock which is assumed to be constant $(0 \leq d \leq 1)$.

Furthermore, we define the following three important ratios.[2]

$y = Y/N =$ average labor productivity which is assumed to grow at the constant rate a (i. e., $\dot{y}/y = a > 0$).[3]

$v = (wN)/(pY) = w/(py)$ = workers' share in national income.

$u = Y/K =$ capital-output ratio, which is also called 'rate of capital utilization'.

Next, let us consider the equations which constitute the basic model in order.

2-1. Workers' consumption

A fraction d ($0 < d < 1$) of workers' income is collected in the form of taxes. Thus their disposable income is $(1-d)vpY$, and we assume that they consume all their disposable income. In this case, we have the following equation.

$$pC_W = (1-d)vpY \tag{1}$$

2-2. Capitalists' consumption

According to Wolfstetter(1982), we assume that only capitalists purchase government bond. For simplicity, we assume that government bond is the 'consol type'. In other words, we assume that a holder of a unit of government bond is paid a unit of money every period by the government. In this case, the nominal interest payment per period becomes B and the market price of government bond becomes the reciprocal of the nominal rate of interest, i. e., $p_B = 1/i$.[4] Furthermore, we assume that the tax base on the capitalists' income is the income *net* of the purchase of the government bond, and the average tax rate on capitalists' income is the same as that on wage income. In this case, capitalists' disposable income becomes $(1-d)\{(1-v)pY + B - p_B\dot{B}\}$ $= (1-d)\{(1-v)pY + B - \dot{B}/i\}$. We assume that capitalists' consumption is proportional to their disposable income. Namely,

$$pC_K = c_k(1-d)\{(1-v)pY + B - \dot{B}/i\} \tag{2}$$

where c_k is capitalists' average propensity to consume ($0 < c_k < 1$).

2-3. Firms' investment and pricing behavior

Next, let us consider firms' behavior with respect to investment and pricing. Contrary to Goodwin(1967)'s original model, we introduce firms' investment function which is independent of the saving function of the capitalists' household according to Keynesian tradition. The following investment function is assumed.

$$I = H(1-v, r)Y \equiv H(1-v, i-p^e)Y \ ; \ H_1 \equiv \partial H/\partial(1-v) > 0, \ H_2 \equiv \partial H/\partial r < 0 \tag{3}$$

where $r = i - p^e$ is the expected real rate of interest of the government bond.

Such an investment function is justified by Kaldor(1961). He emphasizes six 'stylized facts' as a starting point for modeling the process of economic change, and the fifth stylized fact indicates the positive correlation between the share of profits in national income and the share of investment in output. In addition, Eq. (3) says that real investment expenditure is a decreasing function of the expected real rate of interest, which is a standard Keynesian postulate.[5] We also suppose that

$$\dot{K} = I - dK \tag{4}$$

; which simply means that the *net* investment contributes to the changes of the capital stock.

Now, we shall turn to the formulation of firms' pricing behavior. We suppose that firms set a desired price p^D as a constant markup, m, on the unit labor cost (cf. Kalecki(1971)). Namely,

$$p^D = mwN/Y = mw/y. \tag{5}$$

However, the desired price is not always realized. The actual dynamics are governed by

$$\dot{p}/p = g(\dot{p}^D/p^D) \quad ; \quad 0 < g < 1, \tag{6}$$

where $0 < g < 1$ means a sort of price rigidity. For a full account of this point, see, for example, Sportelli (1995, pp. 41-42).

2-4. Government's fiscal policy

The tax revenue of government consists of the income taxes on workers and capitalists, namely,

$$pT = d\{vpY + (1-v)pY + B - \dot{B}/i\} = d(pY + B - \dot{B}/i). \tag{7}$$

Furthermore, we specify government's spending policy as follows.

$$pG = dpY + m(u^* - u)pY \tag{8}$$

The first term of Eq. (8) indicates a regular expenditure which is proportional to national income. The second term represents a discretionary expenditure. We assume that the government accurately recognizes the macroeconomic structure and thus knows the equilibrium rate of capacity utilization u^* which is determined by $u^* = F^{-1}(\alpha)$, where the function $F(u)$ is the wage-Phillips curve which will be introduced later. Government's fiscal policy is counter-cyclical when $\mu > 0$, while it is pro-cyclical when $\mu < 0$. The former is called the 'Keynesian policy rule' and the latter is termed the 'classical rule' by Wolfstetter(1982).

Following Wolstetter(1982), we assume that the government deficit is financed only by selling bond to the capitalists class. Thus the government budget constraint is expressed as follows.[6]

$$\dot{B}/i = B + p(G - T) \tag{9}$$

As for the monetary policy rule of the central bank, we adopt the Post Keynesian 'Horizontalist view' in the sense of Moore(1988). In other words, we assume that the central bank accommodates money supply to the money demand endogenously to keep the nominal rate of interest (i) at some constant level. This means that the nominal money supply is proportional to the nominal national income if we assume the standard Keynesian money demand function $L^D = f(i)pY$. This hypothesis contrasts with that of Asada(1991) and Franke and Asada(1994) which adopt the 'Verticalist view' with constant growth rate of the nominal money supply and endogenous nominal rate of interest.

2-5. Adjustment process in the goods market

We formalize the adjustment process in the goods market as follows.

$$\dot{u} = \varepsilon\{(C_W + C_K + I + G)/K - u\} \quad ; \quad \varepsilon > 0 \tag{10}$$

We assume that the output-capital ratio at the full utilization of capital stock is fixed, but even in this case, the actual output-capital ratio becomes a variable when the capital stock is not fully utilized. In fact, the actual output-capital ratio is proportional to the rate of capital utilization. Eq. (10) says that the capital utilization fluctuates according as the excess demand in the goods market per capital stock is positive or negative. This is a formalization of the Keynesian / Kaleckian quantity adjustment process in the modern capitalist economy, in which the goods market is not always cleared by flexible price adjustment.[7] This feature of our model contrasts with the original Goodwin(1967)'s model in which a sort of 'Say's law' is assumed. In fact, in Goodwin(1967), there is no investment function which is independent of the saving function, and it is assumed that the investment is automatically adjusted to keep the full utilization of capital stock in every period.

It is worth to note that the adjustment process which is expressed by Eq. (10) implicitly assumes that the discrepancy between demand and production is absorbed through the changes of the inventory, and it is supposed that the demand side is always realized.[8] As Chiarella and Flaschel(1996) pointed out correctly, in this type of the formulation, "income concept is based on production plans and not on actual sales"(Chiarella and Flaschel(1996) p. 330).

2-6. Wage adjustment process

We model the wage adjustment process by rather standard expectations-augmented wage Phillips curve :

$$\dot{w}/w = F(u) + \pi^{e} \quad ; \quad F'(u) > 0. \tag{11}$$

This formulation follows the procedure by Franke and Asada(1994). Namely, capital utilization (u) is adopted as a proxy for employment rate or the tightness of labor market. As Franke and Asada(1994) notes, this procedure can save one state variable (employment rate), and "this simplification is justified by the high correlation on the two variables over the cycle" (Franke and Asada(1994) p. 277). Furthermore, we assume the following inequalities as a rather technical requirement.

$$F(0) < \alpha < F(\bar{u}) \tag{12}$$

; where $\bar{u}$ is the output-capital ratio under the full utilization of the capital stock.

2-7. Two missing equations

Eq. (1) through Eq. (11) constitute eleven independent equations with thirteen endogenous variables $(C_W, C_K, I, K, p^D, p, T, G, B, u, w, \pi^e, v)$. Therefore, we need two more equations to close the system.

The first missing equation concerns the expectations formation process of the price inflation. We adopt the following adaptive expectations hypothesis (cf. Asada(1991), Franke and Asada(1994), Chiarella and Flaschel(1996) et. al).

$$\dot{p}^e = b(\dot{p}/p - p^e) \quad ; \quad b > 0 \tag{13}$$

We can obtain the second missing equation by the logarithmic differentiation of the equation $v = w/(py)$, which is nothing but the definition of v, i. e.,

$$\dot{v}/v = \dot{w}/w - \dot{p}/p - a \tag{14}$$

These two equations can close our basic model.

3. Analysis of the basic model without policy lag

In this section, we shall analyze mathematically the performance of the basic model *without policy lag* which was formulated in the previous section.

3-1. Reduced form of equations

We can transform the system in the previous section into the following reduced form, which is a three-dimensional nonlinear dynamical system.

(i) $\dot{u} = e\{H(1-v, i-p^e) - (1-c_k)(1-d)(1-v) + m(1-c_k)(u^*-u)\}u$

$\equiv f_1(u, v, p^e\ ; m)$

(ii) $\dot{v} = (1-g)\{F(u) + p^e - a\}v \equiv f_2(u, v, p^e)$

(iii) $\dot{p}^e = b[g\{F(u) + p^e - a\} - p^e] \equiv f_3(u, p^e)$ $\qquad (S_1)$

We can obtain Eq. (S_1) (i) as follows. Substituting equations (7) and (8) into Eq. (9), we have

$$B - \dot{B}/i = pT - pG = d(B - \dot{B}/i) - m(u^*-u)pY. \tag{15}$$

Solving this equation with respect to $B - \dot{B}/i$, we obtain

$$B - \dot{B}/i = -m(u^* - u)pY/(1-d). \qquad (16)$$

Substituting equations (1), (2), (3), (8) and (16) into Eq. (10), we obtain Eq. (S_1) (i).

On the other hand, we can obtain Eq. (S_1) (ii) as follows. Differentiating Eq. (5) and substituting it into Eq. (6), we have

$$\dot{p}/p = g(\dot{w}/w - \dot{y}/y) = g(\dot{w}/w - a). \qquad (17)$$

Substituting equations (17) and (11) into Eq. (14), we obtain Eq. (S_1) (ii). Finally, Eq. (S_1) (iii) is obtained from equations (11), (13) and (17).

It is worth to note that in our model the dynamics of the government bond do not feed back into the movement of u, v, or p^e so that the system becomes decomposable unless we introduce the wealth effect on the capitalists' consumption.[9]

3-2. The properties of the equilibrium solution

The non-zero equilibrium solution of the system (S_1) is given by $(u^*, v^*, p^e{}^*)$ such that $\dot{u} = \dot{v} = \dot{p}^e = 0$, where

$$H(1-v^*, i) - (1-c_k)(1-d)(1-v^*) = 0, \qquad (18)$$

$$F(u^*) - a = 0, \qquad (19)$$

$$p^e{}^* = 0. \qquad (20)$$

It is obvious from Eq. (11) and the inequality (12) that there exists a unique $u^* \in (0, \bar{u})$. We *assume* the existence of the unique solution $v^* \in (0,1)$. Furthermore, the following assumption is added.

Assumption 1. $H_1{}^* \equiv [\partial H / \partial(1-v)]^* > (1-c_k)(1-d)$.

This means that the marginal propensity to invest of firms exceeds the marginal propensity to save *at the equilibrium point*, which is a standard assumption in Kaldorian business cycle theory (cf. Kaldor(1940), Asada(1987), Asada(1991), and Asada and Yoshida(2001)).

Let us discuss the equilibrium solution from the economic point of view. From Eq. (20), the expected rate of inflation is zero, so that the same is true of actual rate of inflation. What is more, it follows from Eq. (19) that the nominal wage rate grows at growth rate of labor productivity, which means that the real wage rate also grows at the same rate because of no price inflation. The equilibrium rate of economic growth is determined endogenously by the formula

$$\dot{Y}/Y = \dot{K}/K = H(1-v^*, i)u^* - d \equiv g^*. \tag{21}$$

Substituting $u = u^*$ into Eq. (16), we also have

$$\dot{B}/B = i. \tag{22}$$

It follows from equations (21) and (22) that the bond-income ratio (B/Y) eventually tends to zero at the long run equilibrium *if and only if* the inequality

$$i < g^* \tag{23}$$

is satisfied. We *assume* that in fact this inequality is satisfied. In this case, the structure of the government bond is sustainable in the long run.

How can the equilibrium values of the system change when there is change in any of the policy parameters? A change in m has no effect upon the long run equilibrium position, because the target rate of capacity utilization is fixed at the equilibrium rate. A rise in d increases the wage share under **Assumption 1**, i. e.,

$$\partial v^*/\partial d = (1-c_k)(1-d)/\{H_1^* - (1-c_k)(1-d)\} > 0. \tag{24}$$

3-3. Local stability analysis and Hopf bifurcation

To inquire into the local stability of the equilibrium point, we use the coefficients of the linearized system near the equilibrium point. The Jacobian matrix of the system (S_1), which is evaluated *at the equilibrium point*, is given by

$$J_1 = \begin{bmatrix} f_{11}(m) & f_{12} & f_{13} \\ f_{21} & 0 & f_{23} \\ f_{31} & 0 & f_{33} \end{bmatrix} \tag{25}$$

; where $f_{11}(m) = -e(1-c_k)u^*m$, $f_{12} = -e\{H_1{}^* - (1-c_k)(1-d)\}u^* < 0$, $f_{13} = -eH_2{}^*u^* > 0$, $f_{21} = (1-g)F'^*v^* > 0$, $f_{23} = (1-g)v^* > 0$, $f_{31} = bgF'^* > 0$, and $f_{33} = -b(1-g) < 0$.

Then, we can write the characteristic equation of the basic model as follows.

$$\Delta_1(l) \equiv |lI - J_1| = l^3 + a_1 l^2 + a_2 l + a_3 = 0 \tag{26}$$

; where

(i) $a_1 = -traceJ_1 = -\underset{(?)}{f_{11}(m)} - \underset{(-)}{f_{33}} \equiv a_1(m)$,

(ii) $a_2 = \begin{vmatrix} 0 & f_{23} \\ 0 & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11}(m) & f_{13} \\ f_{31} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11}(m) & f_{12} \\ f_{21} & 0 \end{vmatrix}$

$= \underset{(?)}{f_{11}(m)}\underset{(-)}{f_{33}} - \underset{(+)}{f_{13}}\underset{(+)}{f_{31}} - \underset{(-)}{f_{12}}\underset{(+)}{f_{21}} \equiv a_2(m)$,

(iii) $a_3 = -\det J_1 = -\underset{(-)}{f_{12}}\underset{(+)}{f_{23}}\underset{(+)}{f_{31}} + \underset{(-)}{f_{12}}\underset{(+)}{f_{21}}\underset{(-)}{f_{33}} > 0.$ (27)

From these relationships, we obtain the following result.

$$\Psi(m) \equiv a_1 a_2 - a_3 = Am^2 + Bm + D, \tag{28}$$

where $A = -e^2(1-c_k)^2 u^{*2} \underset{(-)}{f_{33}} > 0$, $B = -e(1-c_k)u^*(\underset{(+)}{f_{13}}\underset{(+)}{f_{31}} - \underset{(-)}{f_{12}}\underset{(+)}{f_{21}} - f_{33}^2)$, and

$D = \underset{(+)}{f_{13}}\underset{(+)}{f_{31}}\underset{(-)}{f_{33}} + \underset{(-)}{f_{12}}\underset{(+)}{f_{23}}\underset{(+)}{f_{31}} < 0.$

It follows from Eq. (28) that

$$\Psi(0) = D < 0, \tag{29}$$

which implies that the equilibrium point of the system (S_1) becomes locally unstable because one of the Routh-Hurwitz conditions for stable roots ($a_1 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 > 0$) is violated when the government's fiscal policy is 'neutral' ($m = 0$).

Next, let us investigate how the government can stabilize the potentially unstable economy by means of the local stability analysis. As a preliminary of such an analysis, let us note that the quadratic equation $\Psi(m) = 0$ has two real roots (m_L, m_H) such that $m_L < 0 < m_H$, and we have $\Psi(m) < 0$ for all $m \in (m_L, m_H)$, and we have $\Psi(m) > 0$ for all $m \in (-\infty, m_L) \cup (m_H, +\infty)$.

Proposition 1.

The equilibrium point of the system (S_1) is locally stable for all $m \in (m_H, +\infty)$, and it is locally unstable for all $m \in (-\infty, m_H)$.

Proof.

(i) It is easy to check that all of the Routh-Hurwitz conditions for stable roots are satisfied when $m > m_H$.

(ii) From Eq. (27)(i) we can see that we have $a_1 = 0$ when $m = m_1 \equiv -b(1-g)/\{e(1-c_k)u^*\} < 0$, and $a_1 < 0$ ($a_1 > 0$) is obtained whenever $m < m_1$ ($m > m_1$). This means that the equilibrium point is locally unstable when $m < m_1$ because one of the Routh-Hurwitz conditions is violated. Furthermore, from Eq. (28) we have $\Psi(m_1) = -a_3 < 0$, which means that $m_L < m_1$, and thus we have $\Psi(m) < 0$ for all $m \in [m_1, m_H)$. This implies that the equilibrium point becomes locally unstable also in the case of $m_1 \leq m < m_H$. .

This proposition implies that the government cannot stabilize the potentially unstable economy if the 'classical' fiscal policy rule ($m < 0$) is adopted, but the government can stabilize the economy at least locally by adopting sufficiently active 'Keynesian' policy rule ($m > m_H$). The particular parameter value m_H defines the 'bifurcation point' which divides the parameter values of m into unstable and stable regions. We can easily confirm that at $m = m_H$ the following relationships are satisfied.

(i) $a_1 a_2 - a_3 = 0,$

(ii) $a_2 = a_3 / a_1 > 0$,
(iii) $\partial(a_1 a_2 - a_3)/\partial m > 0$. (30)

These relationships are enough to apply the Hopf bifurcation theorem to establish the existence of the cyclical movement.

Proposition 2.
There exist some non-constant periodic solutions of the system (S_1) at some parameter values m which are sufficiently close to m_H.

Proof.
We can apply the Hopf bifurcation theorem, which asserts the existence of the closed orbit, if we show that (i) the characteristic equation (26) has a pair of pure imaginary roots and a non-zero real root, and (ii) the real part of the imaginary roots is not stationary with respect to the changes of the parameter m.[10] These conditions are equivalent to the following conditions in terms of the coefficients of the characteristic equation : $a_1 a_2 - a_3 = 0$, $a_2 > 0$, and $\partial(a_1 a_2 - a_3)/\partial m \neq 0$ at the bifurcation point.[11] Eq. (30) implies that all of these conditions are in fact satisfied at $m = m_H$. .

4. Time lag in fiscal policy

The argument in the previous section makes it clear that Keynesian fiscal policy has stabilizing effects in our analytical framework of a Keynes-Goodwin growth cycle model. However, we could obtain this result by assuming that the government can respond to the changes of the economic environment *without time lag*. In actual, the effect of stabilization policy depends on the length of policy lag, as Friedman(1948) pointed out in a classical paper. This theme has been repeatedly taken up in many elementary textbooks on Macroeconomics. Obviously, it is doubtful that demand management policies are conducted thoroughly and timely by the policy makers in the real world. The government actually faces limitations on forecasting ability and difficulties in political processes, so that the timing of policy would be subject to delay. Therefore, it will be of great significance that we study the effects of policy lag on the macroeconomic stability.

4-1. A general model of policy lag

Phillips(1954, 1957) argues the above-mentioned subject employing a simple multiplier-accelerator model, while the argument of Friedman(1948) is intuitive and descriptive. Phillips presents three types of fiscal policy : proportional, integral, and derivative stabilization policies.[12] In this subsection, we adopt a sort of integral stabilization policy to examine the effects of time lag explicitly.[13] The government spending function is assumed to have the following form.

$$G(t)/Y(t) = d + m\int_{-\infty}^{t}\{u^* - u(s)\}w(s)ds \qquad (31)$$

; where

$$w(s) = (\frac{n}{t})^n \frac{(t-s)^{n-1}}{(n-1)!} e^{-(n/t)(t-s)} \ , \quad t > 0. \qquad (32)$$

Note that n is a positive integer. If we note that $\int_{-\infty}^{t} w(s)ds = 1,$ the function $w(s)$ can be thought of as a weighting function, which is identical with a density function with the mean, t, and the variance, t^2/n.[14] Thus the economic meaning of Eq. (31) is that the length of policy lag is t on average. When $n = 1,$ it is the exponential distribution. For $n \geq 2,$ $w(s)$ has a one-hump form with a maximum value at $s = t - (n-1)t/n,$ for fixed t. We depict the graphs of $w(s)$ with $t = 1.5$ and $n = 2,$ 16, and 74 in Fig. 1. It shows that a sharp peak appears around $s = t - t$ as n increases. Henceforth, we shall call the parameter t 'policy lag' for simplicity.

Insert Fig. 1 here.

We have an analytical advantage over Phillips(1954, 1957), since our expression (31) enables us to consider the policy lag explicitly, as mentioned above. Our expression is similar to Phillips' in that government expenditures are dependent not on the degree of the capacity utilization at the particular time but on the whole past data on u. However, there is a slight difference between two expressions. That is, we connect the level of government spending with the weighted average of the past sequence of $u^* - u$ by employing the weighting function which is expressed by Eq. (32), whereas, in Phillips' formula, government purchases are made in proportion to the time integral (or, summation) of the past sequence of $u^* - u$ with $w(s) \equiv 1$ on the time interval $(-\infty, t)$. [15]

Using the fact that $\int_{-\infty}^{t} w(s)ds = 1$, we can convert Eq. (31) to the form

$$G(t)/Y(t) = d + m(u^* - \int_{-\infty}^{t} u(s)w(s)ds). \tag{33}$$

Replacing Eq. (8) in section 2 with the new equation (33), we obtain the following reduced form of a Keynes-Goodwin model with policy lag, which is a system of nonlinear integro-differential equations.

$$\begin{aligned}
&\text{(i)} \quad \dot{u}(t) = e\{H(1-v(t), i - p^e(t)) - (1-c_k)(1-d)(1-v(t)) \\
&\qquad\qquad + m(1-c_k)(u^* - \int_{-\infty}^{t} u(s)w(s)ds)\}u(t) \\
&\text{(ii)} \quad \dot{v}(t) = (1-g)\{F(u(t)) + p^e(t) - a\}v(t) \\
&\text{(iii)} \quad \dot{p}^e(t) = b[g\{F(u(t)) + p^e(t) - a\} - p^e(t)] \qquad (S_2)
\end{aligned}$$

At first glance, it seems that this system is so complicated that it is intractable. Fortunately, however, we can transform this system into a relatively tractable system of (nonlinear) ordinary differential equations by using the so called 'linear chain trick', which is due to MacDonald(1978).[16] To this end, let us define

$$x_j(t) = \int_{-\infty}^{t} (\frac{n}{t})^j \frac{(t-s)^{j-1}}{(j-1)!} e^{-(n/t)(t-s)} u(s)ds \quad ; \quad j = 1, 2, \ldots, n. \tag{34}$$

Differentiation of Eq. (34) with respect to time gives us

$$\dot{x}_1(t) = (n/t)\{u(t) - x_1(t)\}, \tag{35}$$

$$\dot{x}_j(t) = (n/t)\{x_{j-1}(t) - x_j(t)\} \quad ; \quad j = 2, \ldots, n. \tag{36}$$

This trick produces the following system of $(n+3)$ dimensional ordinary differential equations.[17]

$$\text{(i)}\quad \dot{u} = e\{H(1-v, i-p^{e}) - (1-c_k)(1-d)(1-v) + m(1-c_k)(u^*-x_n)\}u$$

$$\equiv G_1(u, v, p^{e}, x_n \ ; \ m)$$

$$\text{(ii)}\quad \dot{v} = (1-g)\{F(u) + p^{e} - a\}v \equiv G_2(u, v, p^{e})$$

$$\text{(iii)}\quad \dot{p}^{e} = b[g\{F(u) + p^{e} - a\} - p^{e}] \equiv G_3(u, p^{e})$$

$$\text{(iv)}\quad \dot{x}_1 = (n/t)(u - x_1) \equiv G_4(u, x_1 \ ; \ t)$$

$$\text{(v)}\quad \dot{x}_j = (n/t)(x_{j-1} - x_j) \equiv G_{j+3}(x_{j-1,} x_j \ ; \ t)$$

$$; \quad j = 2, \ldots, n. \qquad (S_2)'$$

The equilibrium solution of this system is essentially identical to that of the system (S_1), namely,

$$H(1-v^*, i) - (1-c_k)(1-d)(1-v^*) = 0, \tag{37}$$

$$F(u^*) - a = 0, \tag{38}$$

$$p^{e*} = 0, \tag{39}$$

$$x_1^* = x_2^* = \ldots = x_n^* = u^*. \tag{40}$$

The Jacobian matrix of this system *at the equilibrium point* becomes as follows.

$$J_2 = \begin{bmatrix} 0 & G_{12} & G_{13} & 0 & 0 & \cdots & 0 & G_{1n+3}(m) \\ G_{21} & 0 & G_{23} & 0 & 0 & \cdots & 0 & 0 \\ G_{31} & 0 & G_{33} & 0 & 0 & \cdots & 0 & 0 \\ n/t & 0 & 0 & -n/t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & n/t & -n/t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -n/t & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & n/t & -n/t \end{bmatrix} \tag{41}$$

; where $G_{12} = -e\{H_1^* - (1-c_k)(1-d)\}u^* < 0$, $G_{13} = -eH_2^* u^* > 0$,

$G_{1n+3}(m) = -e(1-c_k)u^*m$, $G_{21} = (1-g)F'^*v^* > 0$, $G_{23} = (1-g)v^* > 0$, $G_{31} = bgF'^* > 0$, and $G_{33} = -b(1-g) < 0$.

The characteristic equation of this system becomes

$$\Delta_2(l) \equiv |lI - J_2| = 0. \tag{42}$$

First, let us investigate the local stability of this system in case of $m = 0$. In this case, Eq. (42) becomes as follows because of $G_{1n+3}(0) = 0$.

$$\Delta_2(l) = (l + n/t)^n |lI - V| = (l + n/t)^n (l^3 + m_1 l^2 + m_2 l + m_3) = 0, \tag{43}$$

where

$$V = \begin{bmatrix} 0 & G_{12} & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & G_{33} \end{bmatrix} \tag{44}$$

and

$$m_1 = -traceV = -\underset{(-)}{G_{33}} > 0, \tag{45}$$

$$m_2 = \begin{vmatrix} 0 & G_{23} \\ 0 & G_{33} \end{vmatrix} + \begin{vmatrix} 0 & G_{13} \\ G_{31} & G_{33} \end{vmatrix} + \begin{vmatrix} 0 & G_{12} \\ G_{21} & 0 \end{vmatrix} = -\underset{(+)}{G_{13}}\underset{(+)}{G_{31}} - \underset{(-)}{G_{12}}\underset{(+)}{G_{21}}, \tag{46}$$

$$m_3 = -\det V = -\underset{(-)}{G_{12}}\underset{(+)}{G_{23}}\underset{(+)}{G_{31}} + \underset{(-)}{G_{12}}\underset{(+)}{G_{21}}\underset{(-)}{G_{33}} > 0, \tag{47}$$

$$m_1 m_2 - m_3 = \underset{(+)}{G_{13}}\underset{(+)}{G_{31}}\underset{(-)}{G_{33}} + \underset{(-)}{G_{12}}\underset{(+)}{G_{23}}\underset{(+)}{G_{31}} < 0. \tag{48}$$

Eq. (43) implies that in this case the characteristic equation has n multiple roots $l = -n/t < 0$, and other three roots are determined by the equation $|lI - V| = 0$.

Inequality (48) means that the equation $|lI - V| = 0$ has at least one root with positive real part, because one of the Routh-Hurwitz conditions for stable roots is violated. In fact the equation $|lI - V| = 0$ has two roots with positive real parts and one negative real root when three inequalities $m_1 > 0$, $m_3 > 0$, and $m_1 m_2 - m_3 < 0$ are satisfied. [18] Thus, we have proved the following

Proposition 3.

The equilibrium point of the system $(S_2)'$ is locally unstable irrespective of the value of $t \geq 0$ when $m = 0$.

This proposition simply says that the system is unstable when the government's fiscal policy is neutral. On the other hand, the following proposition asserts that the system also becomes unstable if the policy lag is too long even if the government conducts stabilization policy.

Proposition 4.

The equilibrium point of the system $(S_2)'$ becomes locally unstable irrespective of the value of m if the policy lag t is sufficiently large.

Proof.

It is easy to show that

$$\lim_{t \to +\infty} \Delta_2(l) = l^n |lI - V| \tag{49}$$

irrespective of the value of m. This implies, by continuity of the characteristic roots with respect to the changes of the parameter values, that the characteristic equation (42) has at least two roots with positive real parts if t is sufficiently large, because we already know that the equation $|lI - V| = 0$ has two roots with positive real parts and one negative real root.

.

These two propositions are not surprising, and they provide the rigorous foundation to the usual intuitive argument. However, it is difficult to get further outcomes with economic meaning analytically from this general model of policy lag. In the next two subsections, we shall examine two particular cases to establish more accurate results.

4-2. A special case of n = 1

In this subsection, let us consider a special case of $n = 1$. In this case, the system $(S_2)'$ is reduced to the following four-dimentional system of differential equations.

$$
\begin{aligned}
&(\text{ i }) \quad \dot{u} = e\{H(1-v, i-p^e) - (1-c_k)(1-d)(1-v) + m(1-c_k)(u^* - x_1)\}u \\
&\qquad\quad \equiv G_1(u, v, p^e, x_1 \ ; \ m) \\
&(\text{ ii }) \quad \dot{v} = (1-v)\{F(u) + p^e - a\}v \equiv G_2(u, v, p^e) \\
&(\text{ iii }) \quad \dot{p}^e = b[g\{F(u) + p^e - a\} - p^e] \equiv G_3(u, p^e) \\
&(\text{ iv }) \quad \dot{x}_1 = (1/t)(u - x_1) \equiv G_4(u, x_1 \ ; \ t) \qquad (S_3)
\end{aligned}
$$

Although this system is only a special case of the more general system $(S_2)'$ from the mathematical point of view, this simplified version is interesting from the economic point of view, since we can provide clear economic interpretation to this particular adjustment process. We can interpret the variable x_1 as the capacity utilization which is *expected* by the government. Thus, Eq. (S_3) (i) implies that the government's fiscal policy is based on the *expected* capacity utilization. On the other hand, Eq. (S_3) (iv) means that the expected capital utilization changes according to the formula of the adaptive expectations, and the speed of adaptation is the *reciprocal* of the policy lag. The longer the policy lag, the more sluggish is the adaptation.

Proposition 3 and **Proposition 4** also apply to this system, because this system is a special case of the system in section 4-1. In this particular case, however, we can obtain more accurate results analytically.

The Jacobian matrix of the system (S_3) at the equilibrium point becomes

$$
J_3 = \begin{bmatrix} 0 & G_{12} & G_{13} & G_{14}(m) \\ G_{21} & 0 & G_{23} & 0 \\ G_{31} & 0 & G_{33} & 0 \\ 1/t & 0 & 0 & -1/t \end{bmatrix} \qquad (50)
$$

; where $G_{14}(m) = -e(1-c_k)u^* m$, and the definitions of other symbols in the matrix (50) are the same as those in the matrix (41).

We can express the characteristic equation of this system as follows.

$$\Delta_3(l) \equiv \left| lI - J_3 \right| = l^4 + b_1 l^3 + b_2 l^2 + b_3 l + b_4 = 0 \qquad (51)$$

; where

(i) $b_1 = -traceJ_3 = -\underset{(-)}{G_{33}} + 1/t \ > 0,$

(ii) $$b_2 = \begin{vmatrix} 0 & G_{12} \\ G_{21} & 0 \end{vmatrix} + \begin{vmatrix} 0 & G_{13} \\ G_{31} & G_{33} \end{vmatrix} + \begin{vmatrix} 0 & G_{14}(m) \\ 1/t & -1/t \end{vmatrix} + \begin{vmatrix} 0 & G_{23} \\ 0 & G_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 \\ 0 & -1/t \end{vmatrix} + \begin{vmatrix} G_{33} & 0 \\ 0 & -1/t \end{vmatrix}$$

$$= -\underset{(-)}{G_{12}}\,\underset{(+)}{G_{21}} - \underset{(+)}{G_{13}}\,\underset{(+)}{G_{31}} - (1/t)\{\underset{(?)}{G_{14}(m)} + \underset{(-)}{G_{33}}\} \equiv b_2(m),$$

(iii) $$b_3 = -\begin{vmatrix} 0 & G_{23} & 0 \\ 0 & G_{33} & 0 \\ 0 & 0 & -1/t \end{vmatrix} - \begin{vmatrix} 0 & G_{13} & G_{14}(m) \\ G_{31} & G_{33} & 0 \\ 1/t & 0 & -1/t \end{vmatrix}$$

$$- \begin{vmatrix} 0 & G_{12} & G_{14}(m) \\ G_{21} & 0 & 0 \\ 1/t & 0 & -1/t \end{vmatrix} - \begin{vmatrix} 0 & G_{12} & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & G_{33} \end{vmatrix}$$

$$= -\underset{(-)}{G_{12}}\,\underset{(+)}{G_{23}}\,\underset{(+)}{G_{31}} + \underset{(-)}{G_{12}}\,\underset{(+)}{G_{21}}\,\underset{(-)}{G_{33}}$$

$$+ (1/t)\{-\underset{(-)}{G_{12}}\,\underset{(+)}{G_{21}} - \underset{(+)}{G_{13}}\,\underset{(+)}{G_{31}} + \underset{(-)}{G_{33}}\,\underset{(?)}{G_{14}(m)}\} \equiv b_3(m),$$

(iv) $$b_4 = \det J_3 = (1/t)\,\underset{(-)}{G_{12}}(\underset{(+)}{G_{21}}\,\underset{(-)}{G_{33}} - \underset{(+)}{G_{23}}\,\underset{(+)}{G_{31}}) > 0. \qquad (52)$$

Note that the Routh-Hurwitz conditions for local stability in this case become as follows (cf. Gandolfo(1996)).

$$b_j > 0 \ \ (j = 1, 2, 3, 4), \quad b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 > 0 \qquad (53)$$

In this subsection, we shall investigate the local dynamics of the system under the following additional assumption.

Assumption 2.

The value of $|H_2*| \equiv |\{\partial H/\partial(i-p^e)\}*|$ is so small that we have

$$Z \equiv -\underset{(-)}{G_{12}}\underset{(+)}{G_{21}} - \underset{(+)}{G_{13}}\underset{(+)}{G_{31}} > 0.$$

If $|H_2*| = 0,$ we have $G_{13} = 0$ so that Z is positive. Even if $|H_2*| > 0,$ Z becomes positive when the value of $|H_2*|$ is sufficiently small. Assumption 2 implies that the negative impact of the increase of real rate of interest on firms' investment expenditure, which is called 'Mundell effect' and known to be potentially destabilizing, is not very strong *at the equilibrium point*.[19]

Now, we can confirm that the following expression is obtained.

$$\Phi(m) \equiv b_1b_2b_3 - b_1^2b_4 - b_3^2 = Em^2 + Pm + W \quad ; \quad E > 0, \quad W < 0 \qquad (54)$$

where E, P, and W are independent of the parameter m. It is relatively easy to show that

$$E = b(1-g)e^2(1-c_k)^2u*^2/t^3 > 0. \qquad (55)$$

We can also derive the explicit expression of W after tedious calculation, but, fortunately, we can prove that W is in fact negative without such a tedious calculation. The proof is as follows.

When $m = 0,$ we have $b_j > 0$ (j = 1, 2, 3, 4) under **Assumption 2** because $G_{14}(0) = 0.$ Hence, if $W > 0$, all of the Routh-Hurwitz conditions for stable roots (inequalities (53)) are satisfied when $m = 0,$ which contradicts **Proposition 3** which says that the real part of at least one characteristic root becomes positive when $m = 0$. On the other hand, if $W = 0$, the characteristic equation must have a pair of pure imaginary roots and two roots with negative real parts (cf. **Lemma** (ii) in **Appendix**) when $m = 0$, which also contradicts **Proposition 3**. This proves that $W < 0$.

It is obvious from Eq. (54) that we can find two parameter values m_l, m_h such that $\Phi(m_l) = \Phi(m_h) = 0$ and $m_l < 0 < m_h$. Furthermore, we have $\Phi(m) < 0$ for all $m \in (m_l, m_h)$, and we have $\Phi(m) > 0$ for all $m \in (-\infty, m_l) \cup (m_h, +\infty)$.

Proposition 5.

(i) The equilibrium point of the system (S_3) is locally stable for all $m \in (m_h, +\infty)$, and it is locally unstable for all $m \in (-\infty, m_h)$.

(ii) At the point $m = m_h$ a Hopf bifurcation occurs. In other words, there exist some non-constant periodic solutions of the system (S_3) at some parameter values m which are sufficiently close to m_h.

Proof.

(i) It follows from **Assumption 2** that $b_j > 0$ (j = 1, 2, 3, 4) are satisfied for all $m \geq 0$. Therefore, all of the Routh-Hurwitz conditions for stable roots (inequalities (53)) are satisfied when $m > m_h$.

On the other hand, Eq. (52)(ii) implies that there exists a value $m^* < 0$ such that $b_2(m^*) = 0$, and we have $b_2(m) < 0$ ($b_2(m) > 0$) for all $m < m^*$ ($m > m^*$). This means that the system is locally unstable when $m < m^*$. Furthermore, we obtain $\Phi(m^*) = -b_1^2 b_4 - b_3^2 < 0$ at $m = m^*$, which implies that $m_l < m^*$. Therefore, the system becomes locally unstable also in the region $m \in [m^*, m_h)$, because in this region the inequality $\Phi(m) < 0$ is satisfied.

(ii) At the point $m = m_h$, $b_j > 0$ (j = 1, 2, 3, 4) and $\Phi(m_h) = 0$ are satisfied, which implies that the characteristic equation (51) has a pair of pure imaginary roots and two roots with negative real parts (cf. **Lemma** (ii) in **Appendix**). Furthermore, we can easily see that $\Phi'(m) > 0$ at $m = m_h$, which means that the real part of the imaginary roots is not stationary with respect to the changes of m when $m = m_h$. These situations are enough to apply Hopf bifurcation theorem.

This proposition is qualitatively the same as propositions 1 and 2 in section 2. Namely, the sufficiently 'Keynesian' fiscal policy rule can stabilize a potentially unstable economy, and the cyclical movement occurs at the intermediate fiscal parameter values. This conclusion was derived under the assumption that the policy lag is fixed at some level. It must be noted, however, that even if the system is locally stable under some fiscal parameter value when the policy lag is relatively short, the system becomes unstable under the same fiscal parameter value if the policy lag sufficiently increases. This statement follows from **Proposition 4**, which is also applicable to the model in this subsection.

4-3. Limiting case of n ? + 8

Next, let us consider the limiting case of $n \to +\infty$. Recall that the weighting function $w(s)$ which is defined by Eq. (32) is in fact the density function with its mean t and its variance t^2/n. Therefore, the limiting case of $n \to +\infty$ corresponds to the case of the fixed policy lag, $t > 0$. In this case, government behavior is expressed as follows.[20]

$$G(t) = dY(t) + m\{u^* - u(t-t)\}Y(t) \tag{56}$$

Thus, we have the following dynamical system, which is a system of nonlinear differential-difference equations (or delay-differential equations).

$$
\begin{aligned}
&(\text{i}) \quad \dot{u}(t) = e[H(1-v(t), i - p^e(t)) - (1-c_k)(1-d)(1-v(t)) \\
&\qquad\qquad + m(1-c_k)\{u^* - u(t-t)\}]u(t) \equiv G_1(u(t), u(t-t), v(t), p^e(t); m) \\
&(\text{ii}) \quad \dot{v}(t) = (1-g)\{F(u(t)) + p^e(t) - a\}v(t) \equiv G_2(u(t), v(t), p^e(t)) \\
&(\text{iii}) \quad \dot{p}^e(t) = b[g\{F(u(t)) + p^e(t) - a\} - p^e(t)] \equiv G_3(u(t), p^e(t)) \qquad (S_4)
\end{aligned}
$$

The equilibrium point of this system (u^*, v^*, p^{e*} such that $\dot{u}(t) = \dot{v}(t) = \dot{p}^e(t) = 0$ and $u(t) = u(t-t) = u^*$) is also the same as that of the system (S_1). Linearization of this system around the equilibrium point gives the expression

$$(\text{i}) \quad \dot{u}(t) = G_{11}(m)\{u(t-t) - u^*\} + G_{12}\{v(t) - v^*\} + G_{13}\{p^e(t) - p^{e*}\}$$

(ii) $\dot{v}(t) = G_{21}\{u(t) - u^*\} + G_{23}\{p^e(t) - p^{e*}\}$

(iii) $\dot{p}^e(t) = G_{31}\{u(t) - u^*\} + G_{33}\{p^e(t) - p^{e*}\}$ (57)

; where $G_{11}(m) = -e(1 - c_k)u^* m$, and other symbols are the same as those in Eq. (41). Substituting the exponential functions $u(t) - u^* = A_1 e^{lt}$, $v(t) - v^* = A_2 e^{lt}$, and $p^e(t) - p^{e*} = A_3 e^{lt}$ into Eq. (57), we have the following characteristic equation (cf. Bellman and Cooke(1963)).

$$\Delta_4(l) \equiv \left| lI - J_4(l) \right| = 0 \tag{58}$$

where

$$J_4(l) = \begin{bmatrix} G_{11}(m)e^{-tl} & G_{12} & G_{13} \\ G_{21} & 0 & G_{23} \\ G_{31} & 0 & G_{33} \end{bmatrix}. \tag{59}$$

When $m = 0$, we have $J_4(l) = V$, where the matrix V is given by Eq. (44). Therefore, we can conclude that the equilibrium point of the system (S_4) is locally unstable when $m = 0$, because we already know that the equation $\left| lI - V \right| = 0$ has two roots with positive real parts and one negative real root. If $m \neq 0$, Eq. (58) is no longer the simple polynomial but it becomes a transcendental equation, and it has the infinite number of the roots including complex roots.

Although it is difficult to obtain further information which is economically meaningful if we stick to analytical approach, we can obtain some important insight by employing numerical simulations. In the next section, we shall report some results of our numerical simulations of the system (S_4).

5. Numerical simulations of the case of n ? + 8

To practice numerical simulations, we must approximate a system of differential-difference equations (S_4) by a system of difference equations. In this paper, we adopt the following 'Euler's algorithm'.[21]

$$
\begin{aligned}
&(\text{ i }) \quad u(t+\Delta t) = u(t) + (\Delta t)G_1(u(t), u(t-t), v(t), p^e(t)\ \ ; m)\\
&(\text{ ii }) \quad v(t+\Delta t) = v(t) + (\Delta t)G_2(u(t), v(t), p^e(t))\\
&(\text{ iii }) \quad p^e(t+\Delta t) = p^e(t) + (\Delta t)G_3(u(t), p^e(t))
\end{aligned}
\tag{60}
$$

We adopt the time interval $\Delta t = 0.1,$ and the numerical specifications of the involved functions and parameter values are chosen as follows.[22]

$$H(1-v(t), i-p^e(t)) = 1.5(1-v(t))^5 - 0.001(i-p^e(t)) - 0.0027, \tag{61}$$

$$F(u(t)) = 0.1\{1/(1.1-u(t)) - 4.8\}, \tag{62}$$

$$
\begin{aligned}
&a = 0.02, \quad b = 0.8, \quad e = 0.1, \quad g = 0.5, \quad c_k = 0.3,\\
&d = 2/7, \quad i = 0.03, \quad t = 3.1, \quad d = 0.3.
\end{aligned}
\tag{63}
$$

The equilibrium values of u, v, and g are given by

$$u^* = 0.9, \quad v^* = 0.24, \quad g^* = 0.04. \tag{64}$$

Since $i < g^*$, the condition $\lim_{t\to+\infty}\{B(t)/Y(t)\} = 0$ is satisfied at the equilibrium point. The equilibrium values which are given by Eq. (64) are independent of the value of the fiscal parameter m. In other words, the fiscal parameter m cannot affect the long run equilibrium position. This implies that the classical irrelevance theorem seems to apply if we concentrate on the long run equilibrium position. However, the concentration on the long run equilibrium cannot be justified in our model, because in our model the government's fiscal policy can affect the out of steady state dynamics quite drastically, as the following simulation results reveal it clearly.

Figures 2 – 6 are the results of the numerical simulations which correspond to the parameter values $m = 5.00,$ $m = 5.90,$ $m = 5.97,$ $m = 6.06,$ and $m = 6.30$ respectively. All figures are plotted after transient motions are removed from the trajectories.

Insert Fig. 2 – Fig. 6 here.

At the top of each figure, the two-dimensional projection of the trajectory on the (v,u) plane is presented. Furthermore, the bottom of each figure displays the corresponding power spectrum. The abscissa denotes the frequency(Hz) and the ordinate denotes the power spectral density(PSD). As for the exposition of the power spectral analysis, see, for example, Lorenz(1993) chap. 6 and Medio(1992) chap. 5. Lorenz(1993) writes as follows. "A power spectrum can loosely be defined as each frequency's contribution to the overall motion of the time series. .. Power spectra with several distinguishable peaks indicate the presence of quasi-periodic behavior. The dominating peaks represent the basic incommensurable frequencies of the motion, while minor peaks can be explained as linear combinations of the basic frequencies. .. If a continuum of peaks emerges, the power spectrum is said to reflect *broad band noise*. The motion is then either purely random or chaotic for both underlying time concepts." (Lorenz(1993) p. 203) As Medio(1992) noted, "the presence of sharp peaks, however, does not necessarily exclude chaos. Certain embedded periodicities may be present in otherwise chaotic behavior." (Medio(1992) p. 107)

In Figures 2 – 4, we can observe a limit cycle, a period 2 cycle, and a period 4 cycle respectively. Period doubling bifurcations take place. Every bifurcation doubles the number of sharp frequency components. We see that peaks appear in the power spectrum corresponding to submultiples of the fundamental frequency, 0.70 Hz.

Figures 5 and 6 represent chaotic fluctuations. Fig. 5 exhibits narrow-band chaos and Fig. 6 reveals broad-band chaos. They are very similar to the so called Rossler attractor or 'spiral type' chaos (cf. Rossler(1977)). The largest Lyapunov exponents are positive in both cases ; 0.02 and 0.16 respectively.[23] It must be noted, however, that there still remain the sharp peaks or fundamental frequencies in the power spectrum. This type of chaos is often called 'non-mixing chaos' or 'phase coherence' (cf. Crutchfield, Farmer, Packard, Shaw, Jones and Donnelly(1980), and Farmer, Crutchfield, Froehling, Packard and Shaw(1980)).

Insert Fig. 7 – Fig. 8 here.

Fig. 7 is a bifurcation diagram of the variable v with respect to the parameter m, for $2.0 \le m \le 6.3$. This diagram shows the local maxima and minima of v on the business cycles, so that the vertical difference expresses the amplitude of the cycles. Fig. 8 is an enlarged bifurcation diagram, where a period doubling route to chaos is shown clearly and minutely. We can see from Fig. 7 that the stronger application of the Keynesian policy (increase of m) can in fact contribute to stabilize the economy if $m < \overline{m} \approx 4.1$, while more vigorous use of Keynesian policy rather destabilizes the economy (increases the amplitude of the cycles) if $m > \overline{m}$, and the too strong application of the Keynesian policy is responsible for the chaotic movement in this model with fixed policy lag. In this example, the government can establish the minimum amplitude of fluctuation at the parameter value $m = \overline{m}$. Finally, we close this section by making a practical proposal to policy makers. "You should react immediately and promptly to economic disturbances. Otherwise your intended plans for stabilization may cause chaotic business cycles contrary to your intention."

6. Concluding remarks

In this paper, we analyzed basically two models of the policy lag in an analytical framework of the Keynes-Goodwin model of the growth cycle. They are two particular cases of more general model of the distributed policy lag. Unlike Goodwin(1967)'s original model, our models are developed by taking into account Keynesian features : the models emphasize the effective demand. In the first model, it was shown that counter-cyclical fiscal policy is the preferred method for preventing economic fluctuations. Vigorous use of Keynesian policy stabilizes the economy completely. The result of the second model is not simple, however. It gives a good example of controversy on stabilization policy between Keynesians and Monetarists. In case of a short policy lag, a counter-cyclical policy is still available for stabilization. In contrast, if policy is implemented with a long lag, then a very active intervention tends to amplify disturbances and induces the complicated fluctuations in the economy. It is not correct, however, to say that the Keynesian stabilization policy is entirely ineffective to stabilize the potentially unstable economy even in this case. As we observed in the simulation analysis in section 5, the government can reduce the amplitude of the cyclical fluctuation by adopting the proper value of the policy parameter, even if it is impossible to stabilize the economy completely because of the relatively long policy lag.

Appendix

In this appendix, we shall prove the following purely mathematical results, which provide us some useful criteria for the occurrence of Hopf bifurcation in four-dimensional system.

Lemma.

(i) The polynomial equation

$$\Delta(l) \equiv l^4 + b_1 l^3 + b_2 l^2 + b_3 l + b_4 = 0 \tag{A1}$$

has a pair of pure imaginary roots and two roots with non-zero real parts *if and only if* either of the following set of conditions (A) or (B) is satisfied.

(A) $b_1 = 0, \quad b_3 = 0,$ and $b_4 < 0.$

(B) $b_1 \neq 0, \quad b_3 \neq 0, \quad b_4 \neq 0,$ sign b_1 = sign b_3, and

$$\Phi \equiv b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 = 0.$$

(ii) The polynomial equation (A1) has a pair of pure imaginary roots and two roots with negative real parts *if and only if* the following set of conditions (C) is satisfied.

(C) $b_1 > 0, \quad b_3 > 0, \quad b_4 > 0,$ and $\Phi \equiv b_2 b_2 b_3 - b_1^2 b_4 - b_3^2 = 0.$

Proof.

(i) (1) **'If' part**. Suppose that a set of conditions (A) is satisfied. Then, we have

$$\Delta(l) = l^4 + b_2 l^2 + b_4 = 0 \quad ; \quad b_4 < 0. \tag{A2}$$

In this case, we obtain

$$l^2 = \begin{cases} (-b_2 - \sqrt{b_2^2 - 4b_4})/2 \equiv a_1 < 0 \\ (-b_2 + \sqrt{b_2^2 - 4b_4})/2 \equiv a_2 > 0 \end{cases} \tag{A3}$$

so that we have four roots l_j (j = 1, 2, 3, 4) such that

$$l_1, l_2 = \pm i\sqrt{-a_1} \quad , \quad l_3, l_4 = \pm \sqrt{a_2} \qquad \text{(A4)}$$

where $i = \sqrt{-1}.$

Next, suppose that a set of conditions (B) is satisfied. In this case, we can rewrite Eq. (A1) as follows.

$$\Delta(l) = l^4 + b_1 l^3 + (b_1 b_4 / b_3 + b_3 / b_1) l^2 + b_3 l + b_4$$

$$= (l^2 + b_3 / b_1)(l^2 + b_1 l + b_1 b_4 / b_3) = 0 \qquad \text{(A5)}$$

where $b_3 / b_1 > 0,$ $b_1 \neq 0,$ and $b_1 b_4 / b_3 \neq 0$, so that we have four roots such that

$$l_1, l_2 = \pm i\sqrt{b_3 / b_1}, \quad l_3 + l_4 = -b_1 \neq 0, \quad l_3 l_4 = b_1 b_4 / b_3 \neq 0. \qquad \text{(A6)}$$

It is obvious that the real parts of l_3 and l_4 in (A6) are not zero.

(2) '**Only if' part.** Suppose that Eq. (A1) has two roots such that $l_1 = w\ i,$ $l_2 = -w\ i$ ($w \neq 0$) and real parts of other two roots (l_3, l_4) are not zero. Then, Eq. (A1) becomes as follows.

$$\Delta(l) = (l - w\ i)(l + w\ i)(l - l_3)(l - l_4)$$

$$= (l^2 + w^2)\{l^2 - (l_3 + l_4) l + l_3 l_4\}$$

$$= l^4 - (l_3 + l_4) l^3 + (w^2 + l_3 l_4) l^2 - (l_3 + l_4) w^2 l + w^2 l_3 l_4 = 0 \qquad \text{(A7)}$$

In this case, we have the following relationships.

$$b_1 = -(l_3 + l_4), \quad b_2 = w^2 + l_3 l_4, \quad b_3 = -(l_3 + l_4) w^2 = b_1 w^2,$$

$$b_4 = w^2 l_3 l_4, \qquad \text{(A8)}$$

$$\Phi \equiv b_1 b_2 b_3 - b_1^2 b_4 - b_3^2$$

$$= b_1^2 w^2 (w^2 + l_3 l_4) - b_1^2 w^2 l_3 l_4 - b_1^2 w^4 = 0. \qquad \text{(A9)}$$

Only two cases are possible. In the first case, we have $l_3 = -l_4 \neq 0.$ In this case, it follows from (A8) that $b_1 = 0,$ $b_3 = 0,$ and $b_4 < 0.$ In the second case, we have $l_3 \neq -l_4$ and real parts of both roots are not zero. In this case, it follows from (A8) that $b_1 \neq 0,$ $b_3 \neq 0,$

$b_4 \neq 0,$ and sign b_1 = sign b_3.

(ii) (1) **'If' part**. Suppose that a set of condition (C) is satisfied. Then, we have Eq. (A5) with $b_3 / b_1 > 0,$ $b_1 > 0,$ and $b_1 b_4 / b_3 > 0.$ In this case, we have four roots which are given by (A6) with $l_3 + l_4 < 0$ and $l_3 l_4 > 0$. These two inequalities imply that real parts of l_3 and l_4 are negative.

(2) **'Only if' part**. Suppose that Eq. (A1) has a pair of pure imaginary roots $l_1, l_2 = \pm w\, i$ ($w \neq 0$) and real parts of other two roots (l_3, l_4) are negative. In this case, we have the relationships (A8) and (A9) with $l_3 + l_4 < 0$ and $l_3 l_4 > 0$, which imply that a set of conditions (C) is in fact satisfied.

Remarks.

(i) If a set of conditions (A) is satisfied, the condition $\Phi = 0$ is also satisfied.

(ii) If a set of conditions (C) is satisfied, the condition $b_2 > 0$ is also satisfied.

The result (ii) in this lemma provides a complete characterization of the so called 'simple' Hopf bifurcation, while the result (i) provides a complete characterization of the more general types of Hopf bifurcation. In fact, the result (ii) is but a special case of the theorem which was proved by Liu(1994) in a general n-dimensional system. This result is used by Fanti and Manfredi(1998), and it is also implicit in Franke and Asada(1994). Also in the text of this paper, we used only the result (ii) to establish the Hopf bifurcation in our four-dimensional system. Nevertheless, it will be of some interest to report the full content of the **Lemma** with explicit proof because of two reasons. First, our proof can make it explicit that we can derive the result (ii) as a corollary of the more general result (i). Second, as far as we acknowledge, the result (i) of this lemma is not available in usual economic literatures nor textbooks of mathematics, although it provides us some useful information for general type of Hopf bifurcation, which cannot be neglected in some economic models. For example, in dynamic optimization models, we cannot exclude the roots with positive real parts in some situation, so that the knowledge on 'simple' Hopf bifurcation is not enough for the analysis of such a situation.[24]

Notes

(1) Here and henceforth, we suppress the time argument when no confusion is caused. Furthermore, a dot over a symbol denotes the derivative with respect to time.

(2) Definitions of these ratios are the same as those in Franke and Asada(1994).

(3) This assumption means that we are assuming Harrod-neutral exogenous technical progress.

(4) See, for example, Asada(1987).

(5) Eq. (3) also implies that the rate of investment I/K approximately positively correlates with the rate of profit $r=(1-v)Y/K$ and it negatively correlates with the expected real rate of interest. These features are qualitatively similar to the theory of investment which is based on Tobin's q theory (cf. Sargent(1987) and Asada(1989)).

(6) Obviously, Eq. (9) also implies that the government purchases the bond from capitalists when the government budget is in surplus.

(7) Eq. (10) has some similarity with the so called 'Kaldorian' adjustment process, which is formulated as $\dot{Y}=e(C+I+G-Y)$ (cf. Kaldor(1940), Asada(1987)). The formulation of Eq. (10) may be more suitable in the context of the long run growth theory, because in the original Kaldorian formulation, zero excess demand in the goods market is incompatible with economic growth, while Eq. (10) is compatible with the growth equilibrium with zero excess demand (cf. Asada(1991)).

(8) In our formulation as well as the traditional Kaldorian formulation, inventory plays the purely passive role. Obviously, the formulation becomes more complicated if we explicitly introduce the active roles of firms' inventory policy. However, in this paper we do not investigate this theme to concentrate on the analysis of the policy lag. As for the more sophisticated treatment of inventory dynamics, see, for example, Franke (1990, 1996) and Chiarella, Flaschel, Groh and Semmler (2000).

(9) We can observe the similar characteristic also in the so called 'dynamic Keynesian model' which was formulated by Sargent(1987).

(10) See, for example, Gandolfo(1996) or Lorenz(1993).

(11) As for the formal proof, see Asada and Semmler(1995) or Asada(1995).

(12) As for the exposition of Phillips' argument, see, for example, Flaschel(1993) chap. 3.

(13) The policy rule in the previous section is called the proportional stabilization

policy in terms of Phillips.

(14) As for the exposition of this type of the weighting function, see, for example, Medio(1992) chap. 11.

(15) Strictly speaking, in Phillips' formula, the government's target is the deviation of the actual from the desired level of 'output' rather than the degree of capacity utilization. But, this difference is not important here.

(16) As for the examples of the use of the 'linear chain trick' in economic analysis, see, for example, Jarsulic(1993) and Fanti and Manfredi(1998).

(17) In this expression, we suppress the time argument (t) for simplicity of the notation.

(18) As for the proof of this assertion, see Asada(1995).

(19) As for the economic interpretation of the 'Mundell effect', see Chiarella, Flaschel, Groh and Semmler(2000).

(20) This type of the government policy function with fixed policy lag is formulated by Takamasu(1995) and Asada and Yoshida(2001).

(21) As Flaschel(1993) noted, "It(Euler's method) represents the only method which gives rise to a discrete dynamics that can be interpreted in an economically meaningful way"(Flaschel(1993) p. 271).

(22) We also simulated the model by assuming $\Delta t = 0.01$. Even in this case, we obtained virtually the same results. Furthermore, we found that we can obtain the almost same results even if we adopt the Runge-Kutta algorithm instead of the Euler's algorithm.

(23) It is reported in Wolf, Swift, Swinney, and Vastano(1985) that the largest Lyapunov exponent for the Rossler attractor is 0.13.

(24) Feichtinger, Novak and Wirl(1994) provided a set of mathematical conditions for the occurrence of Hopf bifurcation in a special type of the four-dimensional model of dynamic optimization. However, the result (i) of our lemma is more general than their result, because our criteria are applicable to *any* four-dimensional system of differential equations.

Figures

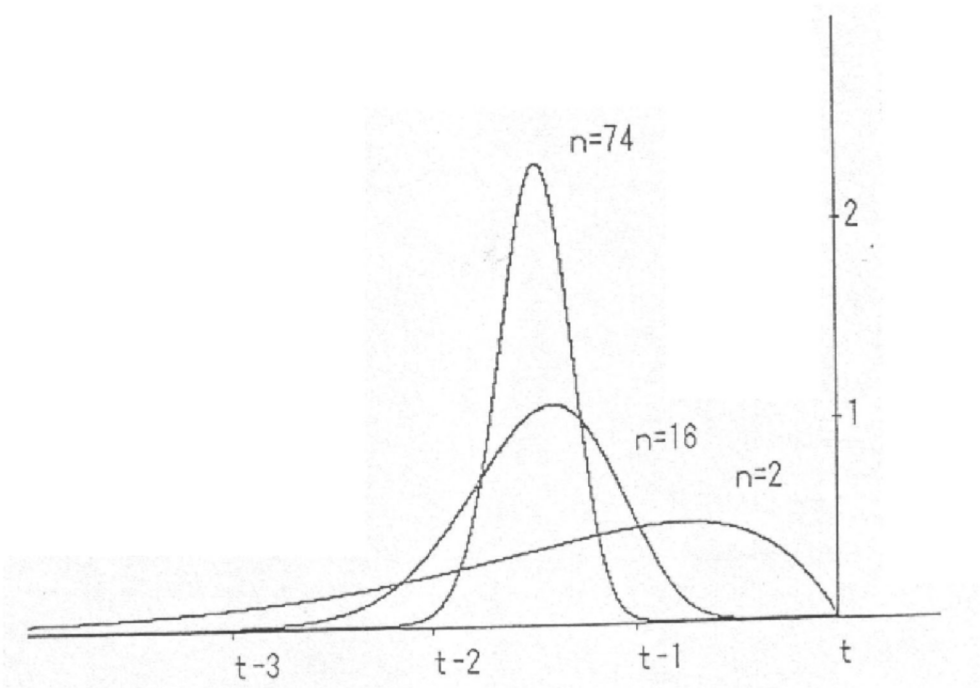


Fig. 1. Weighting function

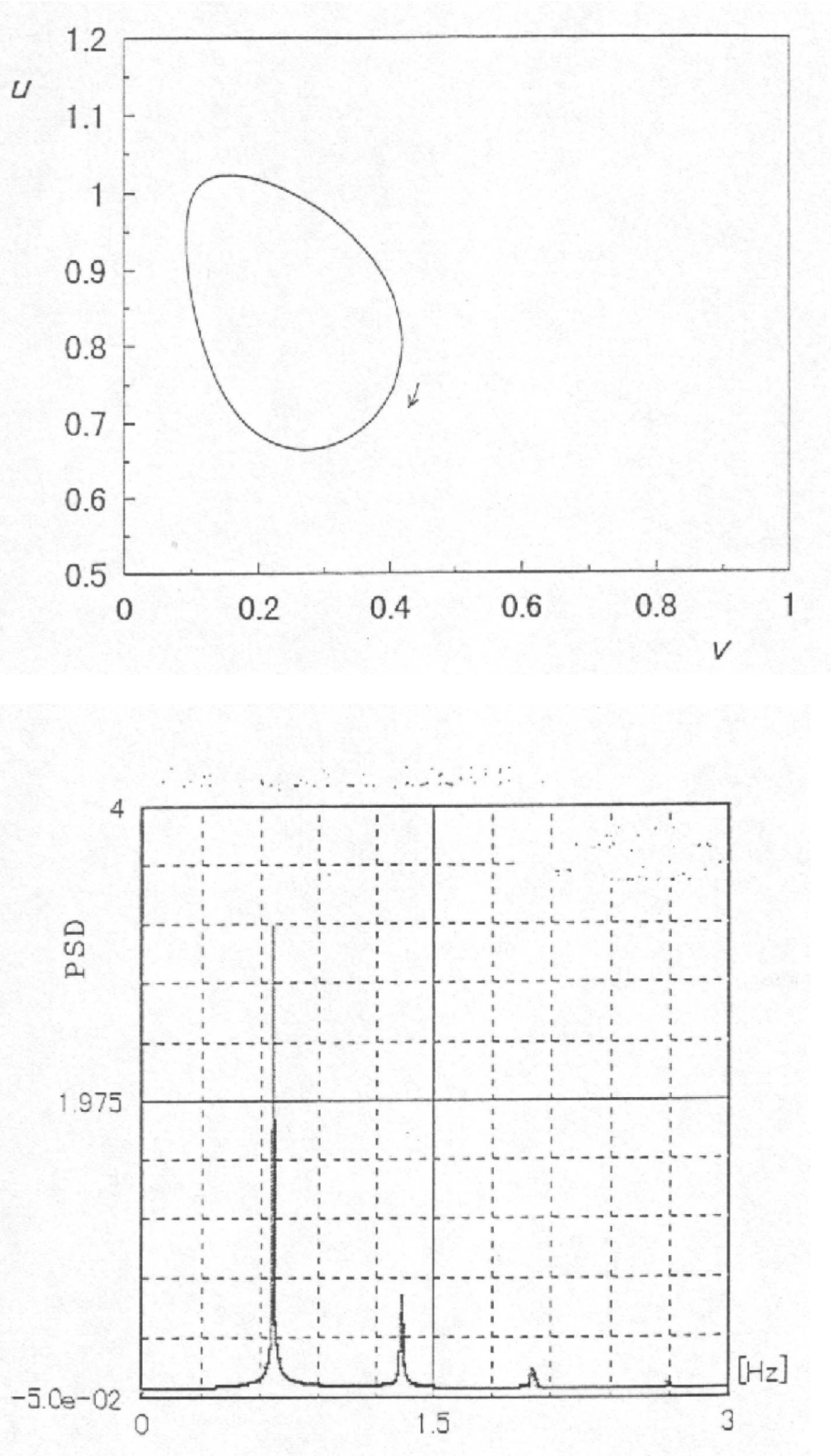
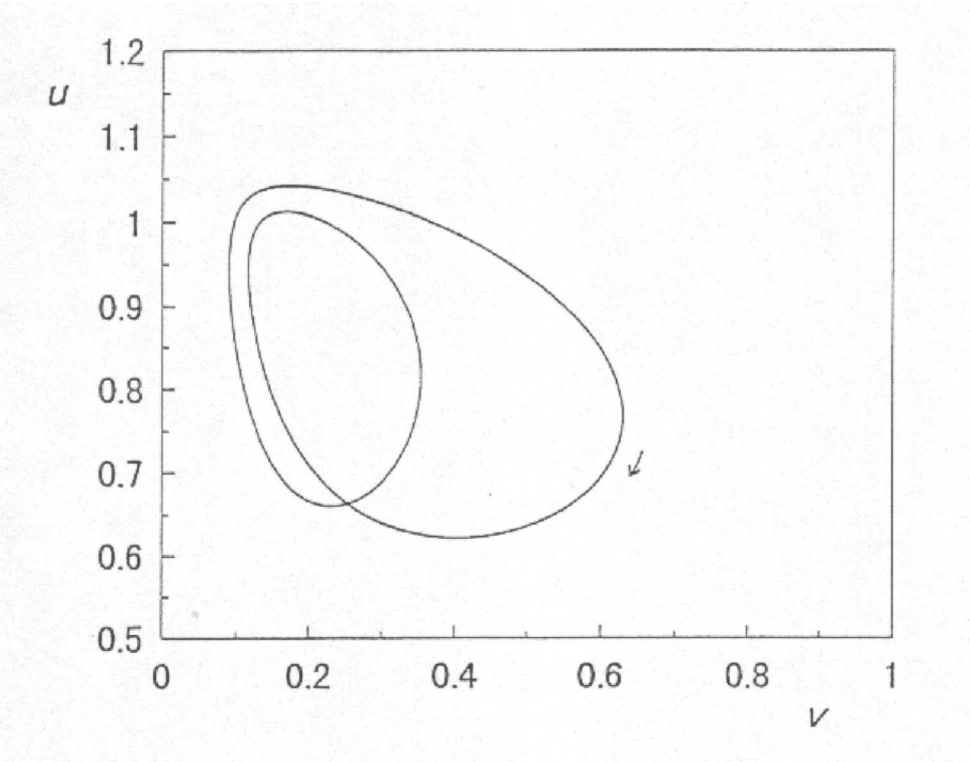


Fig. 2. Limit cycle ($\mu = 5.00$).



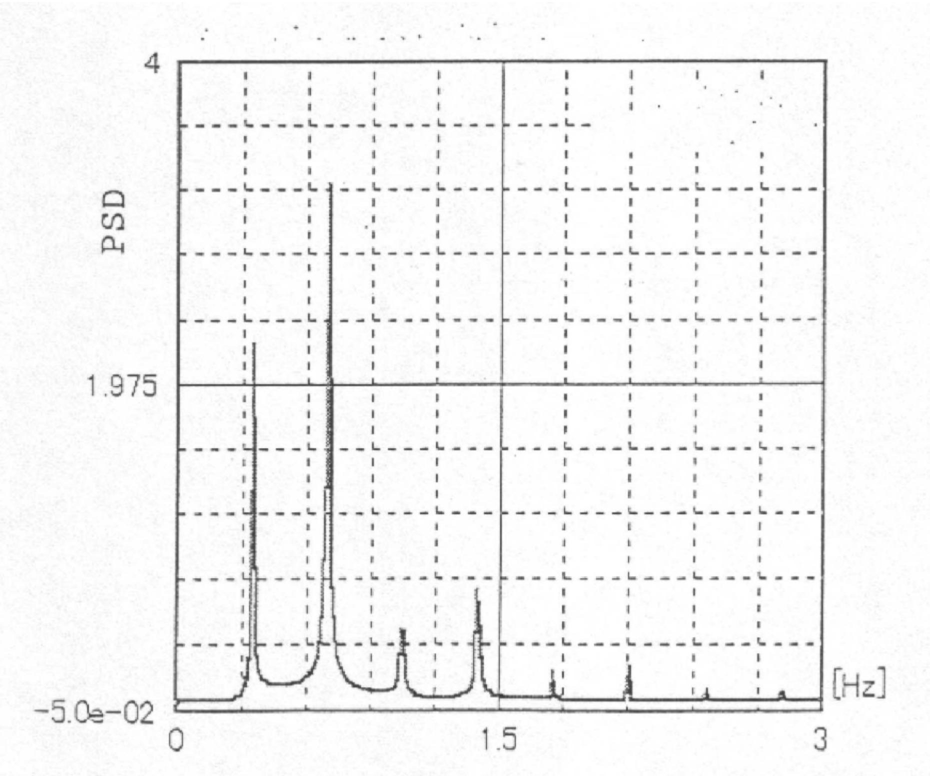


Fig. 3. Period 2 cycle (μ = 5.90).

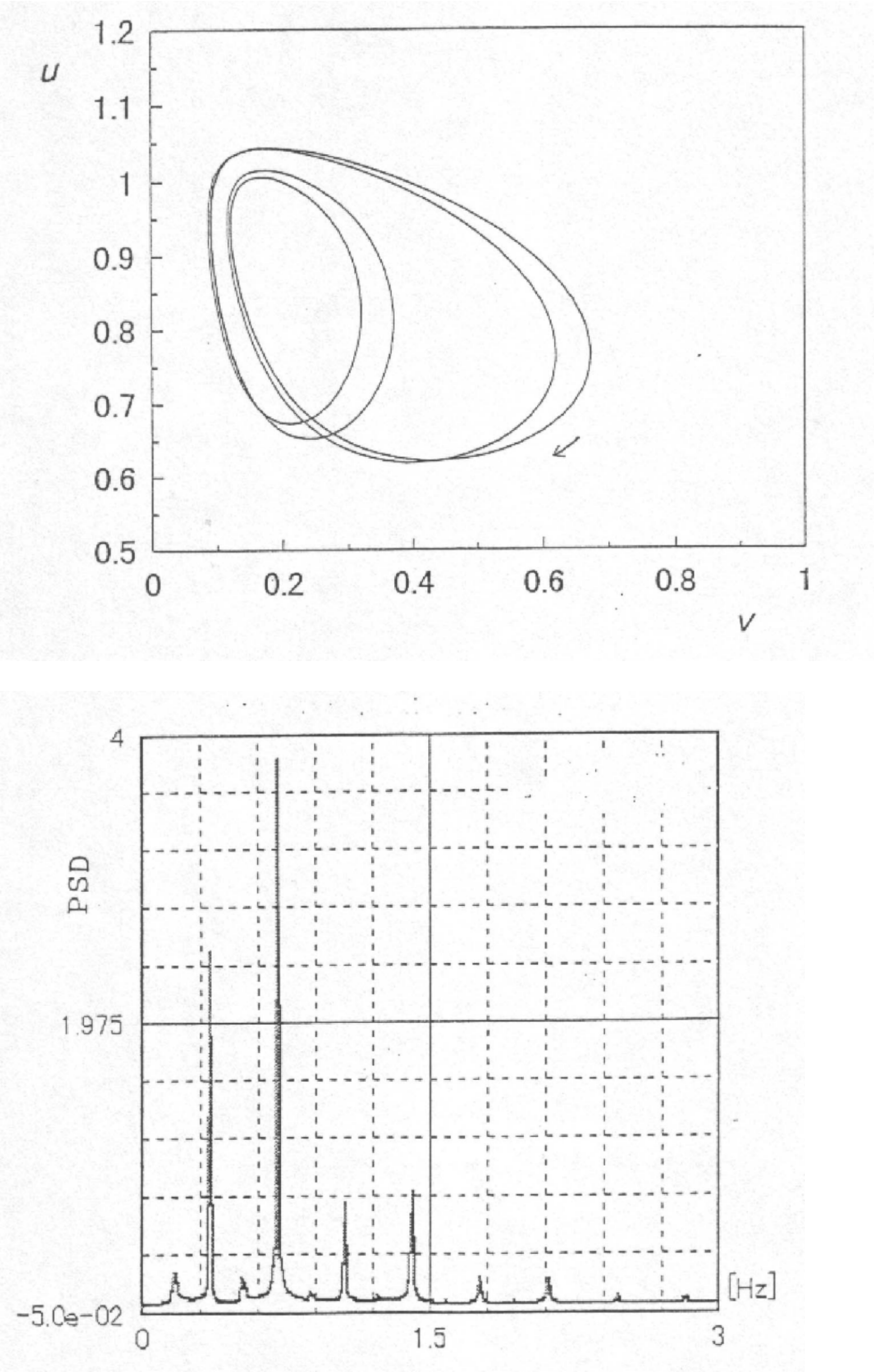


Fig. 4 Period 4 cycle (μ = 5.97).

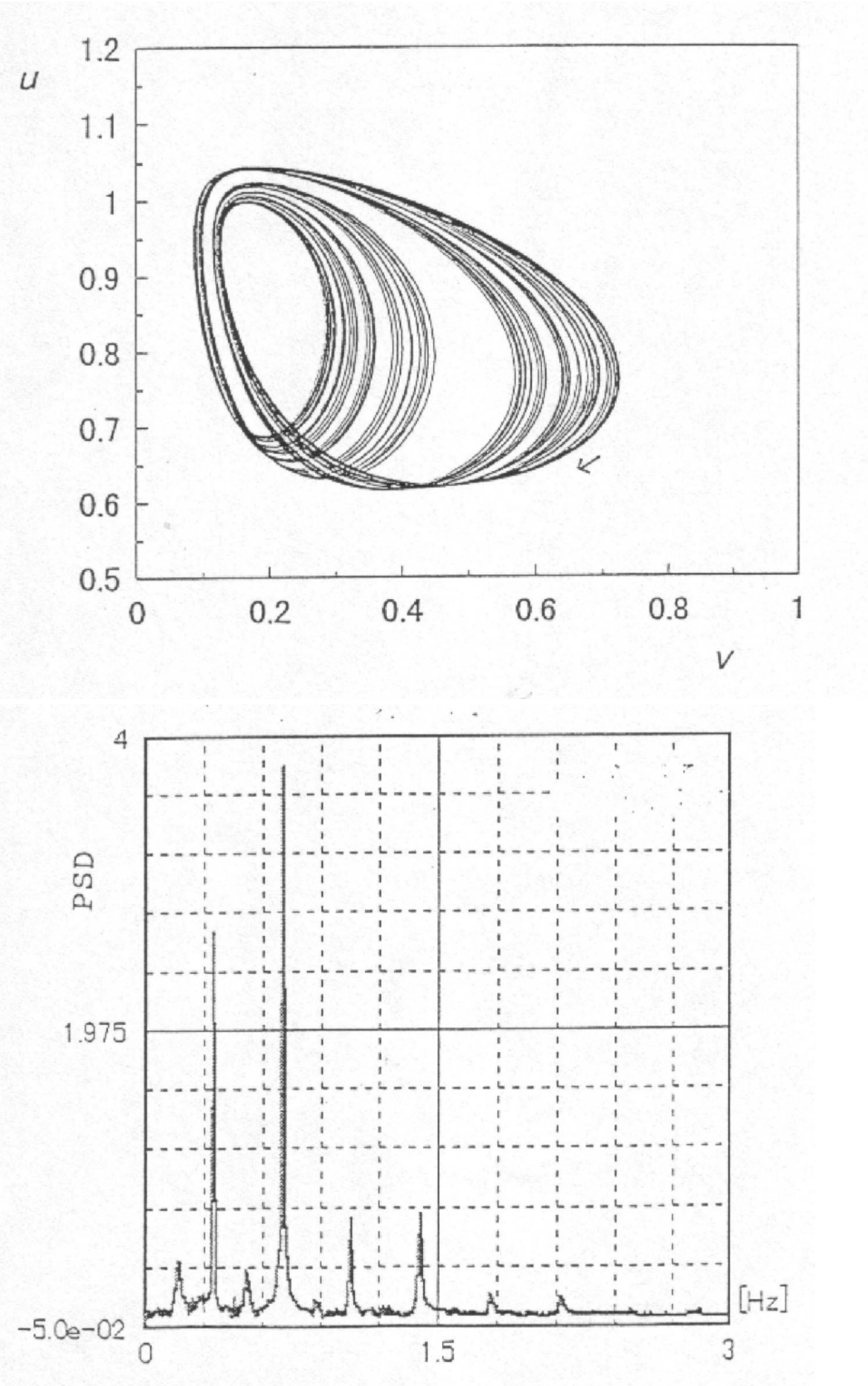


Fig. 5. Narrow-band chaos ($\mu = 6.06$).

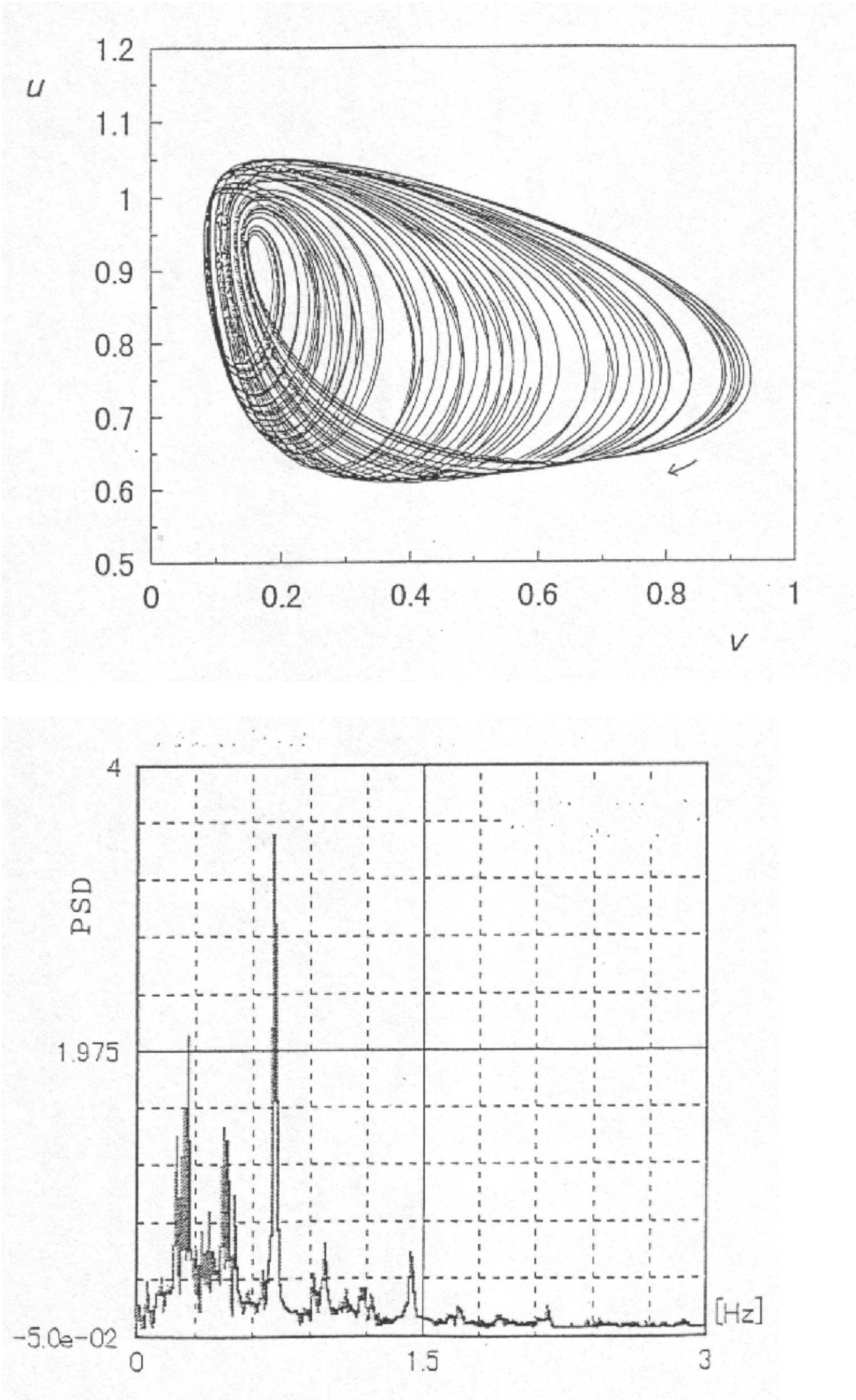


Fig. 6. Broad-band chaos ($\mu = 6.30$).

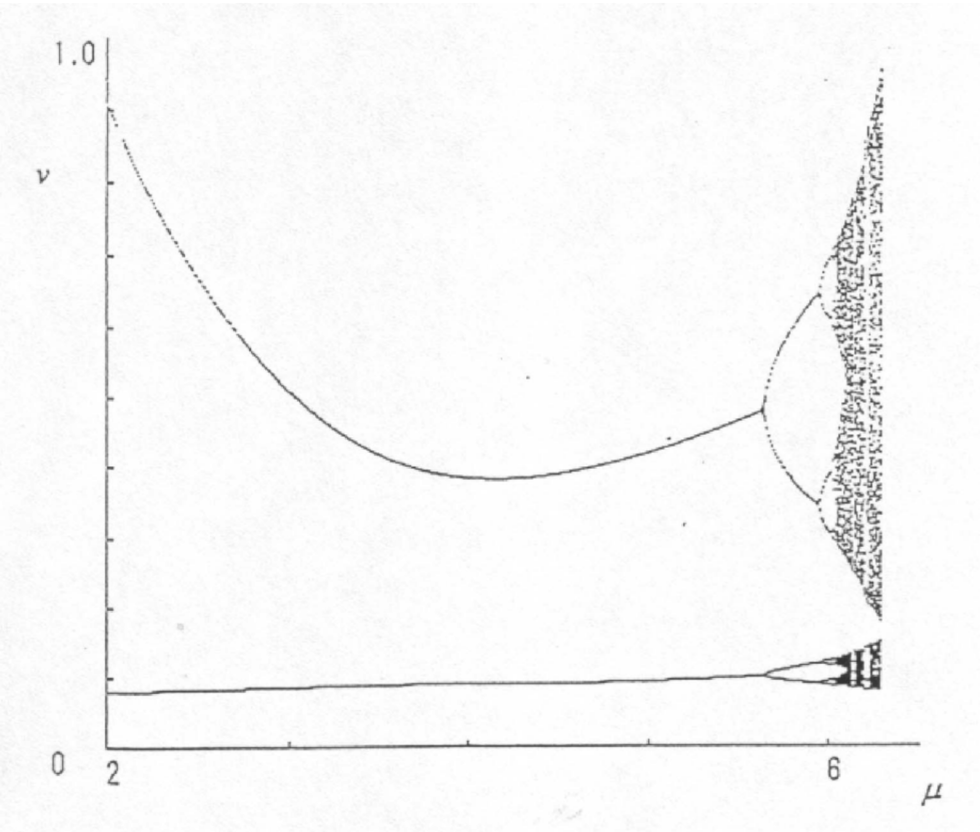


Fig. 7. Bifurcation diagram

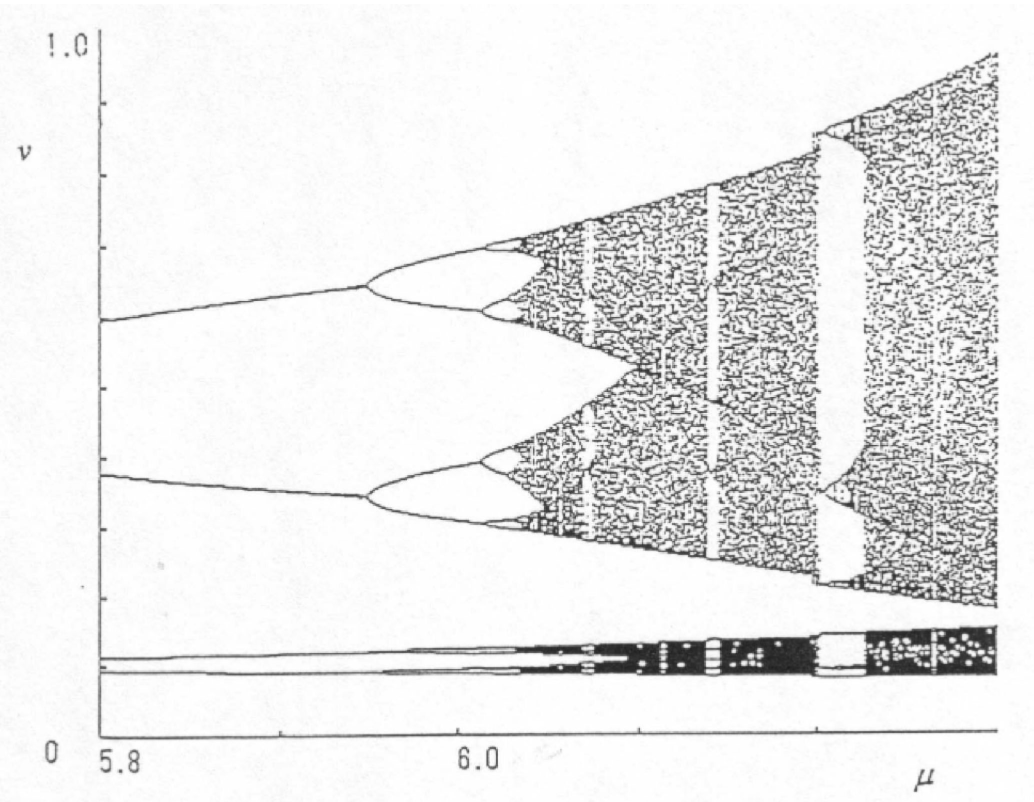


Fig. 8. Bifurcation diagram (an enlarged version).

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