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Limit Pricing and Entry Dynamics with Heterogenous Firms

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Abstract

In this paper we study a dynamic model of pricing and investment with heterogeneous firms under imperfect competition. We assume the existence of two types of firms, a dominant firm and fringe firms. We introduce several asymmetries between the two types of firms. The dominant firm is not financially constrained. It has free access to capital markets although it is subject to increasing adjustment cost of investment. On the other hand, the fringe firms are credit constrained and have no access to capital markets so that they are restricted to internal finance of investment. Furthermore, it is assumed that the dominant firm acts as a price setter and it controls both prices and investment while the fringe firms are price takers who can control only their own investment through internal retention. As in Judd and Petersen (1986), in our model entry is expressed as the growth of the investment of the fringe firms but unlike Judd and Petersen (1986), in our model the trend growth of demand is determined endogenously. The formal structure of our model is described by a dynamic game. More particularly, it represents an open loop Stackelberg differential game of two firms in which the dominant firm acts as a leader and

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fringe firms act passively as followers. We investigate both the steady state as well as the dynamic behavior of the model analytically and numerically.

1 Introduction

Recently, a large number of papers have studied the problem of pricing of firms when there is a threat of entry by other firms. In the literature this pricing behavior has been called limit pricing. In this paper, following Gaskins (1971) and Judd and Petersen (1986) we study an industry where there are two types of firms. In the industry there exist dominant firms which we call for short the dominant firm, and fringe firms, henceforth called fringe firm. We study a dynamic model of pricing, investment and firm valuation for those heterogeneous types of firms under imperfect competition.

We introduce several asymmetries between the two types of firms. First, the dominant firm is not financially constrained. It has free access to capital markets although it is subject to increasing adjustment cost of investment. On the other hand, the fringe firm is credit constrained and has no access to capital markets so that it is restricted to internal finance of investment. Second, it is assumed that the dominant firm acts as a price setter and it controls both prices and investment while the fringe firm is a price taker that can control only its own investment through internal retention. As in Judd and Petersen (1986), in our model entry is expressed as the growth of the investment of the fringe firm but unlike Judd and Petersen (1986), in our model the trend growth of demand is determined endogenously. Third, the formal structure of our model is described by a dynamic game. More particularly, it represents an open loop Stackelberg differential game of two firms in which the dominant firm acts as a leader and fringe firms act passively as followers.

This type of study allows one to explore not only the pricing, investment and entry dynamics in an industry with heterogeneous firms but also, as in Mazzucato and Semmler (1999), the firm valuation and asset price dynamics of heterogeneous firms in an industry the structure of which is evolving over time due to the industry internal dynamics and entry dynamics. Such a study provides one with an approach to explore the asset pricing dynamics of firms as well as an industry from an evolutionary perspective. We investigate both the steady state as well as the dynamic behavior of the model analytically and numerically.

The paper is organized as follows. Section 2 sets up the model, section 3 solves the model by using Pontryagin's maximum principle. Section 4 studies the implied out-of-steady dynamics and section 5 provides some simulations. Section 6 concludes the paper.

2 The Model

Let us define the present value of the dominant firm as follows

$$\begin{aligned} W &= \int_0^{\infty} \{(P_t - c)(D_t - x_t) - \varphi(g_t)K_t\}e^{-\rho t}dt \qquad (1) \\ &= \int_0^{\infty} \{(P_t - c)(E_t - y_t) - \varphi(g_t)\}K_t e^{-\rho t}dt \end{aligned}$$

where P_t, price of the goods, c, average cost of the dominant firm which is assumed to be fixed $(c > 0)$, D_t, real market demand of the goods, x_t, real output of the fringe firm, K_t, real capital stock of the dominant firm, g_t, growth rate of capital stock of the dominant firm $(g_t = \dot{K}_t/K_t)$, ρ, discount rate of the dominant firm which is assumed to be fixed $(\rho > 0)$. Moreover, we presume $\varphi'(0) = 1$, $\varphi''(g_t) > 0$ and denote $E_t = D_t/K_t$, and $y_t = x_t/K_t$,. We assume that the price of the capital good (P_k) is constant, which we normalize so that $P_k = 1$.

We specify the demand function and adjustment cost function as follows.

$$\begin{aligned} D_t = A_t(1 - aP_t); \qquad (2) \\ A_t > 0,\ a > 0,\ 0 \leq P_t \leq 1/a \end{aligned}$$

$$\varphi(g_t) = g_t + \alpha g_t^2;\ \alpha > 0 \qquad (3)$$

Eq. (2) represents a linear downward-sloping demand function, and A_t denotes the 'scale' of the market. We assume that

$$A_t = BK_t;\ B > 0 \qquad (4)$$

We presume that only the investment by the dominant firm contribute to the expansion of the market.[1] Substituting eq. (4) into eq. (2), we have

$$\begin{aligned} E_t = D_t/K_t = B(1 - aP_t); \qquad (5) \\ B > 0,\ a > 0,\ 0 \leq P_t \leq 1/a. \end{aligned}$$

[1] As in Asada and Semmler (1995) we assume that this is achieved through sales strategies, such as advertisement expenditure and built up customer stock.

We suppose that the dominant firm is not financially constrained. It freely can access capital market, but it is subject to increasing adjustment cost, see Uzawa (1969). Eq. (3) is the standard type of adjustment cost function. Substituting eqs. (3) and (5) into eq. (1), we obtain

$$\begin{aligned} W &= \int_0^\infty [(P_t - c)\{B(1 - aP_t) - y_t\} - g_t - \alpha g_t^2] K_t e^{-\rho t} dt \qquad (6) \\ &\equiv \int_0^\infty f(P_t, g_t;\ y_t) K_t e^{-\rho t} dt \end{aligned}$$

where

$$f(P_t, g_t; y_t) \equiv (P_t - c)\{B(1 - aP_t) - y_t\} - g_t - \alpha g_t^2. \qquad (7)$$

Next, let us consider the behavior of the fringe firm. We assume that the fringe firm acts as a price taker. It produces output up to full capacity. In this case, we have

$$\dot{x}_t = \dot{K}_{ft} m;\ m > 0 \qquad (8)$$

where K_{ft} is the capital stock of the fringe firm, and m is the output-capital ratio of the fringe firm in case of full capacity utilization, which is assumed to be constant, see Judd and Petersen (1986). Following Judd and Petersen (1986), we assume that the fringe firm has not free access to capital markets so that the source of finance of the investment of the fringe firm is restricted to its internal finance. Obviously, this is the most strict form of a financial constraint. In this case, we have

$$\dot{K}_{ft} = \overline{s_f} \pi_{ft} = \overline{s_f}(P_t - c_f) x_t, \qquad (9)$$

where $\overline{s_f}$ is the rate of internal retention of the fringe firm (it is assumed that it is fixed at the level $0 < \overline{s_f} < 1$), c_f is the average cost of the fringe firm. We presume that c_f is constant and $c_f \geq c$, and $\pi_{ft} = (P_t - c_f) x_t$ is the profit of the fringe firm. Note that it is assumed that $P_k = 1$, where P_k is the price of the capital good. The expression (9) is meaningful only if $P_t > c_f$ which we will assume henceforth. Substituting eq. (9) into eq. (8), we have

$$\dot{x}_t = \overline{s_f}(P_t - c_f) m x_t. \qquad (10)$$

Therefore, we obtain

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{x}_t}{x_t} - \frac{\dot{K}_t}{K_t} = \overline{s_f}(P_t - c_f)m - g_t \tag{11}$$

so that it follows

$$\dot{y}_t = \{\overline{s_f}(P_t - c_f)m - g_t\}y_t. \tag{12}$$

Although it is assumed that the fringe firm acts as a price taker, s_f must be the control variable of the fringe firm. However, in the appendix, we shall show that the optimal policy of the fringe firm coincides with the policy which maximizes the growth rate $\dot{x}_t/x_t$ so that the optimal retention rate is fixed at the institutionally given upper bound (i.e., the corner solution) under certain assumptions. Because of this reason, we treat $\overline{s_f}$ as a constant through time.

We introduced three asymmetries between the dominant firm and the competitive fringe firm, namely,

1. the dominant firm is a price setter, while the fringe firm is a price taker

2. solely the investment of the dominant firm creates the market (through sales and advertisement expenditure)

3. the dominant firm is not financially constrained but has free access to capital markets, although the investment of the firm is subject to increasing adjustment cost. On the other hand the fringe firm is financially constrained, so that its source of finance is restricted to internal retention of profit

In our model, 'entry' is expressed as the growth of the investment of the fringe firm. As Judd and Petersen (1986) noted, "expansion by the competitive fringe appears to be an important source of 'entry', since full-scale entry by new firms into significant oligopolistic markets appears to be fairly rare event" (pp. 368-369).

To sum up, the optimal problem of the dominant firm is.

$$\max_{P_t, g_t} \int_0^{\infty} f(P_t . g_t; y_t) K_t e^{-\rho t} dt \tag{13}$$

subject to

$$\dot{K}_t = g_t K_t, \quad K_0 > 0 \tag{14}$$

$$\dot{y}_t = \{\overline{s_f}(P_t - c_f)m - g_t\}y_t, \quad y_0 > 0. \tag{15}$$

At a first glance, this is a dynamic optimization problem of the single agent, the dominant firm. However, we can interpret this model as an open-loop Stackelberg differential game between the dominant and fringe firms. In the appendix, we show that the optimal policy of the fringe firm which acts as a follower is to keep $s_f = \overline{s_f}$ for all $t \geq 0$ under some conditions. In this case, the solution of the above problem becomes the solution of the open-loop Stackelberg differential game, in which the dominant firm acts as a leader and the fringe firm as a follower. As for the theory of differential game, see Dockner et al, (2000).

3 The Solution of the Model

We can solve the above problem (13)-(15) by using Pontryagin's maximum principle[2].

The current value Hamiltonain of the above problem (H_t) reads as follows.

$$\begin{aligned} H_t &\equiv f(P_t, g_t;\ y_t)K_t + \lambda_t g_t K_t \\ &+ \mu_t\{\overline{s_f}(P_t - c_f)m - g_t\}y_t \\ &= [(P_t - c)\{B(1 - aP_t) - y_t\} - g_t - \alpha g_t^2]K_t \\ &+ \lambda_t g_t K_t + \mu_t\{\overline{s_f}(P_t - c_f)m - g_t\}y_t. \end{aligned} \tag{16}$$

where λ_t and μ_t are two costate variables which correspond to two dynamic constraints eq. (14) and eq. (15). A set of the optimal conditions for the dominant firm reads as follows.

$$\max_{P_t, g_t} H_t \ \text{ for all } \ t \geq 0. \tag{17}$$

$$\dot{\lambda}_t = -\frac{\partial H_t}{\partial K_t} + \rho\lambda_t \ \text{ for all } \ t \geq 0. \tag{18}$$

[2]Chiang (1992)

$$\dot{\mu}_t = -\frac{\partial H_t}{\partial y_t} + \rho\mu_t \quad \text{for all} \quad t \geq 0. \tag{19}$$

$$\lim_{t\to\infty} \lambda_t e^{-\rho t} = 0 \tag{20}$$

$$\lim_{t\to\infty} \mu_t e^{-\rho t} = 0 \tag{21}$$

We have

$$\begin{aligned} \frac{\partial H_t}{\partial P_t} &= [B(1-aP_t) - y_t - a(P_t - c)B]K_t + \mu_t \overline{s_f} m y_t \\ &= [-2aBP_t - y_t + (1+ac)B]K_t + \mu_t \overline{s_f} m y_t \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial H_t}{\partial g_t} &= [-1 - 2\alpha g_t]K_t + \lambda_t K_t - \mu_t y_t \\ &= [-(1+2\alpha g_t) + \lambda_t]K_t - \mu_t y_t. \end{aligned} \tag{23}$$

$$\frac{\partial^2 H_t}{\partial P_t^2} = -2aBK_t < 0 \tag{24}$$

$$\frac{\partial^2 H_t}{\partial g_t^2} = -2\alpha K_t < 0 \tag{25}$$

$$\frac{\partial^2 H_t}{\partial g_t \partial P_t} = \frac{\partial^2 H_t}{\partial P_t \partial g_t} = 0 \tag{26}$$

From eqs (24) - (26) we have

$$H_{11} < 0, \quad H_{22} < 0,$$

$$\begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} = \begin{vmatrix} H_{11} & 0 \\ 0 & H_{22} \end{vmatrix} = H_{11}H_{22} > 0 \tag{27}$$

where $H_{11} \equiv \frac{\partial^2 H_t}{\partial g_t \partial P_t}$, $H_{22} \equiv \frac{\partial^2 H_t}{\partial g_t^2}$, $H_{12} \equiv \frac{\partial^2 H_t}{\partial g_t \partial P_t}$, $H_{21} \equiv \frac{\partial^2 H_t}{\partial P_t \partial g_t}$.

Therefore, the first order conditions for the maximization of H_t[3].

$$[-2aBP_t - y_t + (1+ac)B]K_t + \mu_t \overline{s_f} m y_t = 0 \tag{28}$$

$$[-[1+2\alpha g_t) + \lambda_t]K_t - \mu_t y_t = 0 \tag{29}$$

Solving eq.(28) with respect to P_t, we obtain

$$\begin{aligned} P_t &= \frac{(-K_t + \mu_t \overline{s_f}\ m)y_t + (1+ac)BK_t}{2aBK_t} \\ &= \frac{(-1+\kappa_t \overline{s_f}\ m)y_t + (1+ac)B}{2aB} \\ &= \frac{1}{2aB}(-1+\kappa_t \overline{s_f}\ m)y_t + \frac{1}{2a} + \frac{c}{2} \end{aligned} \tag{30}$$

where $\kappa_t \equiv \mu_t / K_t$.
Solving eq. (29) with respect to λ_t, we have

$$\lambda_t = 1 + 2\alpha g_t + \kappa_t y_t. \tag{31}$$

On the other hand, eq. (18) and eq. (19) become

$$\begin{aligned} \dot{\lambda}_t = (P_t - c)\{B(-1+aP_t) + y_t \\ + g_t + \alpha g_t^2\} + (\rho - g_t)\lambda_t \end{aligned} \tag{32}$$

$$\dot{\mu}_t = (P_t - c)K_t + \{\rho + g_t - \overline{s}_f(P_t - c_f)m\}\mu_t \tag{33}$$

From eq. (33) we have

$$\begin{aligned} \frac{\dot{\mu}_t}{\mu_t} &= (P_t - c)\frac{K_t}{\mu_t} + \rho + g_t - \overline{s}_f(P_t - c_f)m \\ &= \frac{P_t - c}{\kappa_t} + \rho + g_t - \overline{s}_f(P_t - c_f)m \end{aligned} \tag{34}$$

so that we obtain

[3]The second order conditions are in fact satisfied because of the inequalities (27)

$$\frac{\dot{\kappa}_t}{\kappa_t} = \frac{\dot{\mu}_t}{\mu_t} - \frac{\dot{K}_t}{K_t} = \frac{\dot{\mu}_t}{\mu_t} - g_t \tag{35}$$
$$= \frac{P_t - c}{\kappa_t} + \rho - \overline{s}_f(P_t - c_f)m$$

Eq. (35) implies that

$$\dot{\kappa}_t = P_t - c + \{\rho - \overline{s}_f(P_t - c_f)m\}\kappa_t \tag{36}$$

Next, differentiating eq. (31) with respect to time, we have

$$\dot{\lambda}_t = 2\alpha\dot{g}_t + \kappa_t\dot{y}_t + y_t\dot{\kappa}_t \tag{37}$$

Substituting equations (31) and (37) into eq. (32), we obtain

$$\dot{g}_t = \frac{1}{2\alpha}[(P_t - c)\{B((-1 + aP_t) + y_t + g_t + \alpha g_t^2\} \tag{38}$$
$$+(\rho - g_t)(1 + 2\alpha g_t + \kappa_t y_t) - \kappa_t\dot{y}_t - y_t\dot{\kappa}_t]$$

Substituting eq. (30) into equations (15), (36) and (38), we arrive at the following nonlinear three-dimensional dynamic system

(i)

$$\dot{y}_t = [\overline{s}_f\Big\{\frac{1}{2aB}(-1 + \kappa_t\overline{s}_f\ m)y_t + \frac{1}{2a} + \frac{c}{2} - c_f\Big\}m - g_t]y_t$$
$$\equiv F_1(y_t, \kappa_t, g_t)$$

(ii)

$$\dot{\kappa}_t = \frac{1}{2aB}(-1 + \kappa_t\overline{s}_f\ m)y_t + \frac{1}{2a} - \frac{c}{2} \tag{39}$$
$$+ [\rho - \overline{s}_f\Big\{\frac{1}{2aB}(-1 + \kappa_t\overline{s}_f\ m)y_t + \frac{1}{2a} + \frac{c}{2} - c_f\Big\}m]\kappa_t$$
$$\equiv F_2(y_t, \kappa_t; \rho)$$

(iii)

$$\begin{aligned}
\dot{g}_t &= \frac{1}{2\alpha}[\Big\{\frac{1}{2aB}(-1+\kappa_t\overline{s}_f\ m)y_t + \frac{1}{2a} - \frac{c}{2}\Big\}\Big\{(1+\frac{\kappa_t\overline{s}_f\ m}{2}y_t) \\
&+ \frac{1}{2}(-1-B+caB) + g_t + \alpha g_t^2\Big\} \\
&+ ((\rho - g_t)(1+2\alpha g_t + \kappa_t y_t) \\
&- \kappa_t F_1(y_t, g_t.\kappa_t) - y_t F_2(y_t, \kappa_t; \rho)] \\
&\equiv F_3(y_t, \kappa_t, g_t; \rho)
\end{aligned}$$

Eq. (39) is a system of the fundamental dynamic equations for our model. Next, let us consider the equilibrium solution (y^*, κ^*, g^*) of the system (39) which satisfies $y^* \neq 0$. We can obtain such a solution by solving the following system of the simultaneous equations.

(i)

$$\overline{s}_f\Big\{\frac{1}{2aB}(-1+\kappa\ \overline{s}_f\ m)y + \frac{1}{2a} + \frac{c}{2} - c_f\Big\}m - g = 0$$

(ii)

$$\begin{aligned}
&\frac{1}{2aB}(-1+\kappa\overline{s}_f m)y + \frac{1}{2a} - \frac{c}{2} \\
&+[\rho - \overline{s}_f\Big\{\frac{1}{2aB}(-1+\kappa\ \overline{s}_f\ m)y + \frac{1}{2a} + \frac{c}{2} - c_f\Big\}m]\kappa = 0
\end{aligned} \quad (40)$$

(iii)

$$\begin{aligned}
&\Big\{\frac{1}{2aB}(-1+\kappa\ \overline{s}_f\ m)y + \frac{1}{2a} - \frac{c}{2}\Big\} \\
&\Big\{(1+\frac{\kappa\overline{s}_f\ m}{2})y + \frac{1}{2}(-1-B+caB) + g + \alpha g^2\Big\} \\
&+(\rho - g)(1+2\alpha g + \kappa y) = 0
\end{aligned}$$

The economically meaningful equilibrium solution must satisfy the following conditions.

(i)

$$c_f < P^* < 1/a$$

(ii)

$$0 < y^* < B(1 - aP^*) \quad (41)$$

(iii)

$$\rho > g^*$$

where

$$P^* = \frac{1}{2aB}(-1 + \kappa^* \ \overline{s}_f \ m)y^* + \frac{1}{2a} + \frac{c}{2} \tag{42}$$

Inequality (41)(i) implies that the fringe firm can earn positive profit at equilibrium. If this inequality is satisfied, we also have the inequality

$$c < P^* < 1/a \tag{43}$$

because of the assumption $c_f \geq c$. Inequality (43) means that the dominant firm's profit is also positive at equilibrium. Inequality (41) (ii) implies that the dominant firm and the fringe firm coexist at equilibrium. Inequality (41) (iii) ensures that the net cash flow of the dominant firm (W) becomes finite at equilibrium.

In fact, it follows from eq. (6) that

$$0 < W = \frac{f(P^*, g^*; y^*)}{\rho - g^*} K_0 < +\infty \tag{44}$$

at equilibrium if the inequalities (41) (i) $\sim$ (iii) are satisfied.

Moreover, from eq. (15) we obtain

$$g^* = \overline{s}_f (P^* - c_f) m. \tag{45}$$

Therefore, we get

$$g^* > 0 \tag{46}$$

if the inequality (41)(i) is satisfied. Substituting eq. (45) into eq. (36) and considering $\dot{\kappa}^* = 0$, we obtain

$$\kappa^* = -\frac{P^* - c}{\rho - g^*} \tag{47}$$

Therefore, we have

$$\kappa^* < 0 \tag{48}$$

if the inequalities (41)(i) and (41)(iii) are satisfied.

Remark 1.

It follows from eq. (31) that the quilibrium value of λ_t becomes

$$\begin{aligned} \lambda_t^* &= 1 + 2\alpha g^* + \kappa^* y^* \qquad (49) \\ &= \text{constant}. \end{aligned}$$

Therefore, eq. (20), the 'transversality condition' with respect to λ_t, is in fact satisfied at the equilibrium. On the other hand, the equilibrium value of μ_t becomes

$$\mu_t^* = \kappa^* K_t^* = \kappa^* K_0 e^{g^* t}. \qquad (50)$$

Therefore, we have

$$\lim_{t\to\infty} \mu_t^* e^{-\rho t} = \kappa^* K_0 e^{-(\rho - g^*)t} = 0 \qquad (51)$$

so that the transversality condition with respect to μ_t is also satisfied at equilibrium if the inequality (41) (iii) is satisfied.

Next, let us consider how to solve the system of equations (40). We can rewrite eq. (40) (ii) as follows.

$$A_1(y)\kappa^2 + A_2(y)\kappa + A_3(y) = 0 \qquad (52)$$

where

$$A_1(y) \equiv -\overline{s}_f^2\ m^2 y, \qquad (53)$$

$$A_2(y) \equiv \frac{\overline{s}_f m}{2aB} y + \rho - \overline{s}_f \Big\{ -\frac{1}{2aB} y + \frac{1}{2a} + \frac{c}{2} - c_f \Big\} m, \qquad (54)$$

$$A_3(y) \equiv -\frac{1}{2aB} y + \frac{1}{2a} - \frac{c}{2}. \qquad (55)$$

Next, we state explicitly the following assumptions.

Assumption 1

$$c \leq c_f < 1/a$$

In fact, it is impossible to satisfy the inequality (41)(i) if $c_f \geq 1/a$. We have $A_1,(y) < 0$ for all $y > 0$, and we have $A_3(y) > 0$ if and only if $0 < y < aB(1/a - c) = B(1 - ac)$.

The inequalities (41)(ii) and (43) imply that

$$0 < y^* < B(1 - aP^*) < B(1 - ac). \tag{56}$$

This inequality implies that only the region of y which satisfies $A_3(y) > 0$ is economically meaningful. Therefore, we have only to consider the case of $A_1(y) < 0$ and $A_3(y) > 0$. The solutions of eq. (52) with respect to κ for given $y \in (0, B(1 - ac))$ are as follows

$$\kappa = \begin{cases} \frac{-A_2(y) - \sqrt{\{A_2(y)\}^2 - 4A_1(y)\ A_3(y)}}{2A_1(y)} < 0 \\ \frac{-A_2(y) + \sqrt{\{A_2(y)\}^2 - 4A_1(y)\ A_3(y)}}{2A_1(y)} > 0 \end{cases} \tag{57}$$

We already know that only the solution such that $\kappa < 0$ is economically meaningful, so that we must select the solution such that

$$\kappa(y) \equiv \frac{-A_2(y) - \sqrt{\{A_2(y)\}^2 - 4A_1(y)\ A_3(y)}}{2A_1(y)} < 0 \tag{58}$$

for $y \in (0, B(1 - ac))$, where $A_1(y) < 0$ and $A_3(y) > 0$.

Next, solving eq. (40)(i) with respect to g, we obtain

$$g = \overline{s}_f \Big\{ \frac{1}{2aB}(-1 + \kappa \overline{s}_f m)y + \frac{1}{2a} + \frac{c}{2} - c_f \Big\} m. \tag{59}$$

Substituting eq. (58)into eq. (59), we have

$$g = \overline{s}_f \Big\{ \frac{1}{2aB}(-1 + \kappa(y) \overline{s}_f m)y + \frac{1}{2a} + \frac{c}{2} - c_f \Big\} m \equiv g(y). \tag{60}$$

Substituting eq. (58) and eq. (60) into eq. (40)(iii), we obtain the following equation with the single unknown, y.

$$\begin{aligned} \Phi(y) \equiv [\frac{1}{2aB}\{-1 + \kappa(y)\overline{s}_f\ m\}y + \frac{1}{2a} - \frac{c}{2}] \\ [\Big\{1 + \frac{\kappa(y)\overline{s}_f m}{2}\Big\}y + \frac{1}{2}(-1 - B + caB) + g(y) + \alpha g(y)^2] \\ +\{\rho - g(y)\}\{1 + 2\alpha g(y) + \kappa(y) y\} = 0 \end{aligned} \tag{61}$$

Solving this equation with respect to y, we have the equilibrium solution y^*. Substituting $y = y^*$ into equations (57) and (59), we have the equilibrium values $\kappa^* = \kappa(y^*)$ and $g^* = g(y^*)$. There may be the multiple solutions, but we must neglect the solutions which do not satisfy the inequalities (41)(i) $\sim$ (iii).

4 The Dynamics

Next, let us consider the out of steady state dynamics around the equilibrium point by assuming that the economically meaningful equilibrium exists. The Jacobian matrix of the system (39) which is evaluated at the equilibrium point can be written as

$$J = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & 0 \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \tag{62}$$

where

$$F_{11} \equiv \left(\frac{\partial F_1}{\partial y}\right)^* = \frac{\overline{s}_f^2\ m}{2aB}\ \underset{(-)}{\kappa^*} y^* < 0,$$

$$F_{12} \equiv \left(\frac{\partial F_1}{\partial \kappa}\right)^* = \frac{\overline{s}_f^2\ m}{2aB}\ y^{*2} > 0,$$

$$F_{13} \equiv \left(\frac{\partial F_1}{\partial g}\right)^* = -y^* < 0,$$

$$F_{21} \equiv \left(\frac{\partial F_2}{\partial y}\right)^* = \frac{\overline{s}_f\ m}{2aB}\ \underset{(-)}{\kappa^*}(1 - \overline{s}_f m \underset{(-)}{\kappa^*}) < 0,$$

$$\begin{aligned} F_{22} \equiv \left(\frac{\partial F_2}{\partial \kappa}\right)^* &= \frac{\overline{s}_f\ m}{2aB} y^*(1 - \overline{s}_f m \underset{(-)}{\kappa^*}) \\ &+ \left[\rho - \overline{s}_f\left\{\frac{1}{2aB}(-1 + \kappa^* \overline{s}_f m) y^* + \frac{1}{2a} + \frac{c}{2} - c_f\right\} m\right] \\ &= \frac{\overline{s}_f\ m}{2aB} y^*(1 - \overline{s}_f m \underset{(-)}{\kappa^*}) + \underbrace{(\rho - g^*)}_{(+)} > 0, \end{aligned}$$

$$
\begin{aligned}
F_{31} \equiv \Big(\frac{\partial F_3}{\partial y}\Big)^* \;&=\; \frac{1}{2\alpha}\Big[\frac{1}{2aB}\Big(-1+\underset{(-)}{\kappa^*}\overline{s}_f m\Big)\Big\{\Big(1 \\
&+\; \frac{\kappa^*\overline{s}_f m}{2}\Big)y^* + \frac{1}{2}(-1-B+caB) \\
&+\; g^* + \alpha g^{*2}\Big\} + \Big\{\frac{1}{2aB}\Big(-1+\underset{(-)}{\kappa^*}\overline{s}_f m\Big)y^* \\
&+\; \frac{1}{2a} - \frac{c}{2}\Big\}\Big(1+\frac{\overline{s}_f m}{2}\underset{(-)}{\kappa^*}\Big) \\
&+\; (\rho \underset{(+)}{-} g^*)\underset{(-)}{\kappa^*} - \frac{\overline{s}_f^2 m}{2aB}\kappa^{*2}y^* - \frac{\overline{s}_f m}{2aB}\kappa^*\Big(1-\overline{s}_f m\underset{(-)}{\kappa^*}\Big)y^*\Big]
\end{aligned}
$$

$$
\begin{aligned}
F_{32} \equiv \Big(\frac{\partial F_3}{\partial \kappa}\Big)^* \;&=\; \frac{1}{2\alpha}\Big[\frac{\overline{s}_f\ m}{2aB}y^*\Big\{\Big(1+\frac{\overline{s}_f m}{2}\underset{(-)}{\kappa^*}\Big)y^* \\
&+\; \frac{1}{2}(-1-B-caB) + g^* + \alpha g^{*2}\Big\} \\
&+\; \Big\{\frac{1}{2aB}(-1+\kappa^*\overline{s}_f m)y^* + \frac{1}{2a} - \frac{c}{2}\Big\}\Big\{\frac{\overline{s}_f m}{2}y^*\Big\} \\
&+\; (\rho \underset{(+)}{-} g^*)y^* - \frac{\overline{s}_f^2 m}{2aB}\underset{(-)}{\kappa^*}\Big)y^{*2} \\
&-\; \frac{\overline{s}_f m}{2aB}y^{*2}\Big(1-\overline{s}_f m\underset{(-)}{\kappa^*}\Big) - (\rho \underset{(+)}{-} g^*)y^*\Big],
\end{aligned}
$$

$$
\begin{aligned}
F_{33} \equiv \Big(\frac{\partial F_3}{\partial g}\Big)^* \;&=\; \frac{1}{2\alpha}\Big[\Big\{\frac{1}{2aB}(-1+\kappa^*\overline{s}_f m)Y^* + \frac{1}{2a} \\
&-\; \frac{c}{2}\Big\}\{1+2\alpha g^*\} - \Big(1+2\alpha g^* + \underset{(-)}{\kappa^*}y^*\Big) \\
&+\; (\rho \underset{(+)}{-} g^*)2\alpha + \underset{(-)}{\kappa^*}y^*\Big].
\end{aligned}
$$

The characteristic equation of this system becomes

$$
\Delta(z) \equiv \mid zI - J \mid = z^3 + b_1\ z^2 + b_2 z + b_3 = 0 \tag{63}
$$

where

$$b_1 = -\mathrm{trace} J = -F_{11} - F_{22} - F_{33} \tag{64}$$

$$\begin{aligned} b_2 &= \begin{vmatrix} F_{22} & 0 \\ F_{32} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \\ &= F_{22}F_{33} + F_{11}F_{33} - F_{13}F_{31} + F_{11}F_{22} - F_{12}F_{21} \end{aligned} \tag{65}$$

$$\begin{aligned} b_3 &= -detJ = -F_{11}F_{22}F_{33} - F_{13}F_{32}F_{21} \\ &+ F_{13}F_{22}F_{31} + F_{12}F_{21}F_{33} \\ b_1 b_2 - b_3 &= (-F_{11} - F_{22} - F_{33})(F_{22}F_{33} + F_{11}F_{33} \\ &- F_{13}F_{31} + F_{11}F_{22} - F_{12}F_{21}) + F_{11}F_{22}F_{33} \end{aligned} \tag{66}$$

$$+F_{13}F_{32}F_{21} - F_{13}F_{22}F_{31} - F_{12}F_{21}F_{33} \tag{67}$$

It is expected that under a wide range of parameter sets, the equilibrium point will become a saddle point. In this case, we must select the convergent path because the divergent path will not satisfy the transversality conditions (equations (20) and (21)).

However, there may be some parameter set under which the Hopf-Bifurcation occurs. We can make use of the following useful criterion

Theorem

Hopf-Bifurcation occurs in the system (39) at $\rho = \rho_0 > g^*$ if and only if a set of conditions

$$\begin{aligned} & b_1(\rho_0) \neq 0, \quad b_2(\rho_0) > 0, \\ & b_1(\rho_0) b_2(\rho_0) - b_3(\rho_0) = 0 \end{aligned} \tag{68}$$

and

$$\frac{\partial \{b_1(\rho) b_2(\rho) - b_3(\rho)\}}{\partial \rho} \Big|_{\rho=\rho_0} \neq 0 \tag{69}$$

are satisfied. In this case, three characteristic roots of eq. (62) become

$$z = \begin{cases} i\sqrt{b_2(\rho_0)} \\ -i\sqrt{b_2(\rho_0)} \\ -b_1(\rho_0) \end{cases} \tag{70}$$

where $i = \sqrt{-1}$.

(Proof.)

See Asada (1995) and Asada and Semmler (1995).

If the conditions (68) and (69) are satisfied at $\rho = \rho_0 > g^*$, there exist some non-constant periodic solutions at some parameter values ρ which are sufficiently close to ρ_0. We selected the discount rate ρ of the dominant firm as a bifurcation parameter following the usual procedure of the dynamic optimization theory, yet we could also select another parameter as a bifurcation parameter.

At the closed orbit, y_t, κ_t and g_t are bounded so that λ_t is also bounded from eq. (31). Therefore, at the closed orbit we have $\lim_{t\to\infty} \lambda_t e^{-\rho t} = 0$, which implies that the transversality condition with respect to λ_t (eq. (20) is satisfied at the closed orbit. Furthermore, we have

$$\mu_t = \kappa_t K_t = \kappa_t K_0 e^{\int_0^t g_\tau d_\tau} \tag{71}$$

so that we have

$$\lim_{t\to\infty} \mu_t e^{-\rho t} = \lim_{t\to\infty} (\kappa_t K_0) e^{-\int_0^t (\rho - g_\tau) d_\tau} = 0 \tag{72}$$

if $\rho > g_\tau$ for all $\tau \geq 0$ at the closed orbit. In fact, the closed orbits which satisfy $\rho > g_\tau$ for all $\tau \geq 0$ exist for the parameter values ρ which are sufficiently close to ρ_0 when $\rho > g^*$ at $\rho = \rho_0$.

5 Numerical Simulations

For the convenience of the numerical study let us adopt the transformation

$$\tilde{K}_t \equiv K_t e^{-g^* t} \tag{73}$$

where g^* is the equilibrium growth rate which is endogenously determined by the condition $\dot{y}_t = \dot{\kappa}_t = \dot{g}_t = 0$ in eq. (39). Then the optimization problem which is given by equations (13), (14), and (15) is reduced to the following equivalent expressions

$$\max_{P_t, g_t} \int_0^\infty (P_t, g_t; y_t)\tilde{K}_t e^{-(\rho - g^*)t} dt \tag{74}$$

subject to

$$\dot{\tilde{K}}_t = (g_t - g^*)\tilde{K}_t, \quad \tilde{K}_0 > 0 \tag{75}$$

$$\dot{y}_t = \{\overline{s}_f(P_t - c_f)m - g_t\}y_t, \quad y_0 > 0 \tag{76}$$

Needless to say, at the equilibrium point of the system (15) such that $\dot{\tilde{K}}_t = 0$, we have $g_t = g^*$. We assume that $\rho > g^*$, which is in fact satisfied in our numerical examples. We undertake a numerical study of the above system (74)-(76) by employing a dynamic programming algorithm developed by Grüne (1997).[4]

Periodic Solutions With $\overline{s_f}$=0.3, B=7.3, c=1.38, $c_f = 1.40$, α=11.50, m=0.3, a=0.1, ρ=0.12, we get the economically meaningful equilibrium solution

$$\kappa^* = -0.276, y^* = 6.09, g^* = 0.0012,$$

which satisfies system (41). With starting values of y_t and $\widetilde{K}_t$ being 6.0 and 0.2,the simulations are presented in Figure 1a-1c. From Figures 1a-1c we can observe that for the above parameters periodic solutions arise.

[4]The dynamic programming algorithm of Grüne (1997) was programmed in "Maple" and is available upon request from the authors.

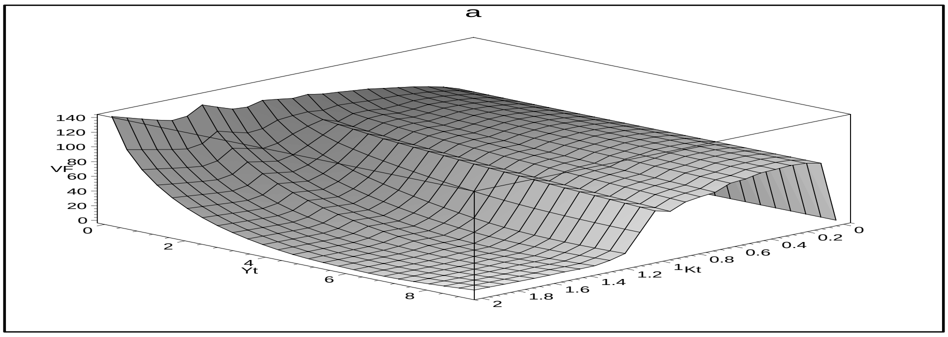

Figure 1a: Value function (Y_t and K_t on the horizontal axes stand for y_t and $\tilde{K}_t$ respectively)

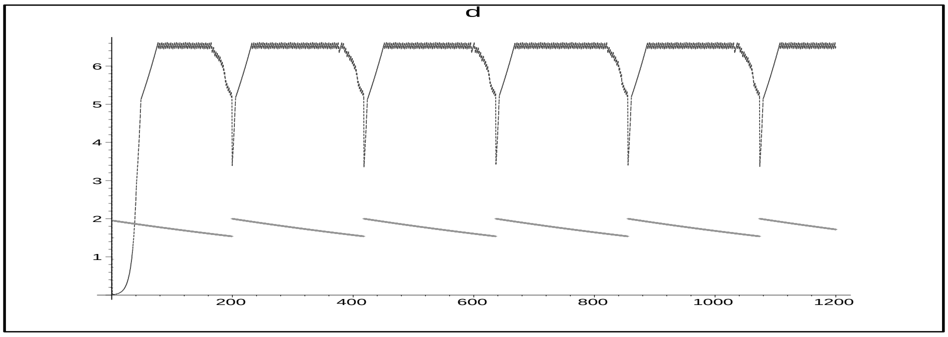

Figure 1b: Paths of y_t (upper part) and $\tilde{K}_t$ (lower part)

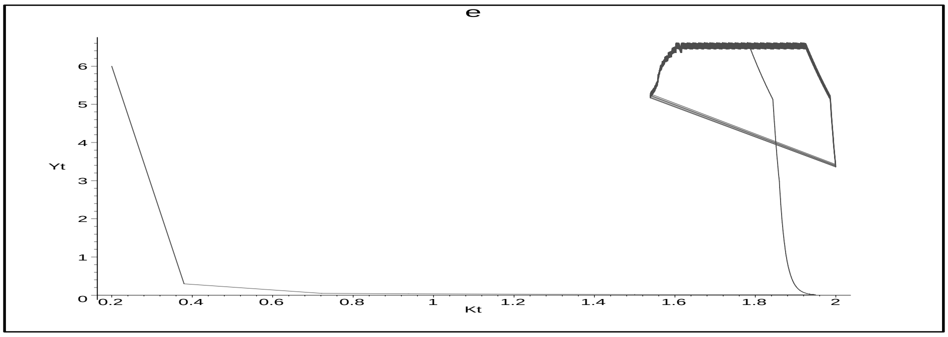

Figure 1c: Phase diagram of y_t (vertical) and $\tilde{K}_t$ (horizontal)

Convergent solutions With $\overline{s_f}$=0.1, B=0.3, c=0.38, $c_f = 0.40$, α=4.50, m=0.3, a=0.3, ρ=0.12, we get the economically meaningful equilibrium solution

$$\kappa^* = -1.65, y^* = 0.22, g^* = 0.005,$$

which satisfies system (41). With the starting values of y_t and $\widetilde{K}_t$ being 0.8 and 2, the simulations are presented in Figure 2a-2c. We can observe that for this second set of parameters converging solutions arise.

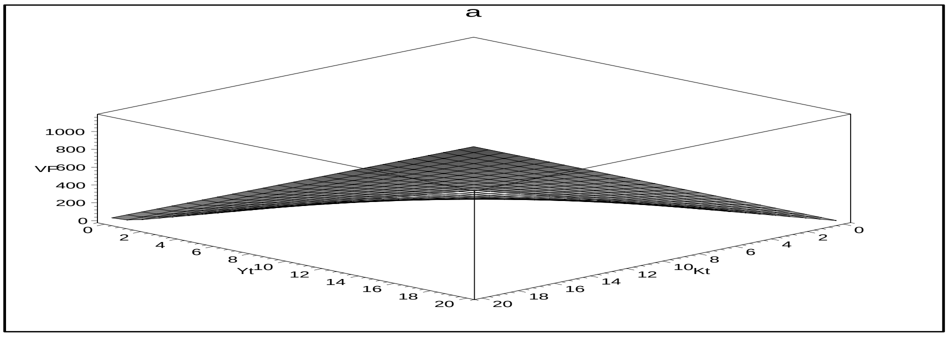

Figure2a: Value function (Y_t and K_t on the horizontal axes stand for y_t and $\tilde{K}_t$ respectively)

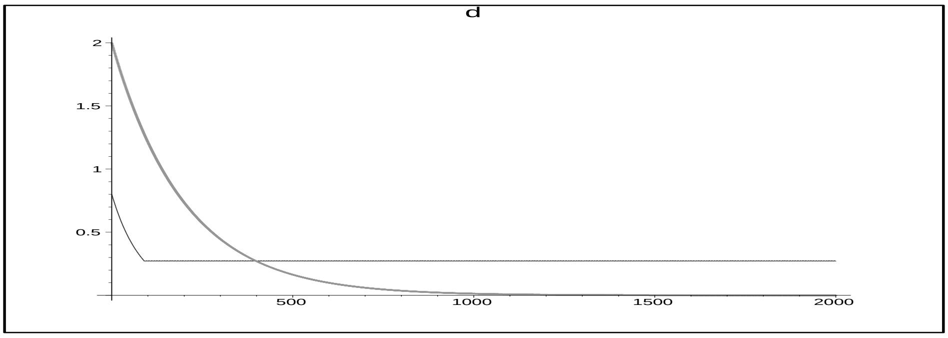

Figure 2b: Paths of y_t (starting value 0.8) and $\tilde{K}_t$ (starting value 2)

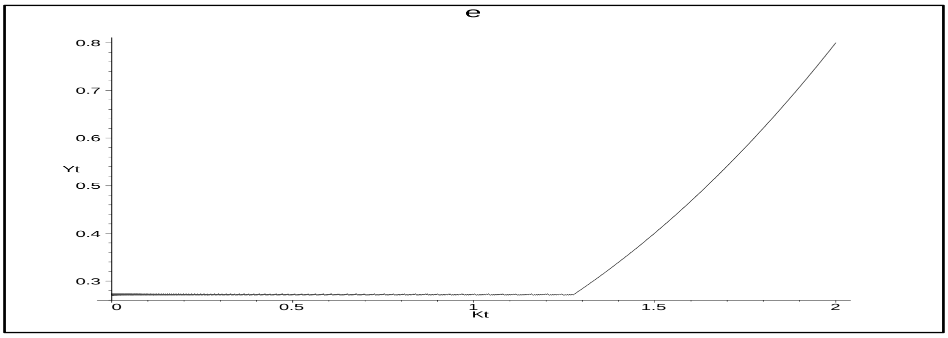

Figure 2c: Phase diagram of y_t (vertical) and $\tilde{K}_t$ (horizontal)

6 Conclusions

In this paper we have studied a model of pricing and investment with heterogenous firms. More specifically we study the interaction of a dominant firm and a fringe firm. The dominant firm that controls the price and its investment is financially unconstrained. The fringe firm that enters the industry and adjusts its output is financally constrained. The trend growth in demand is determined endogenously. The formal structure of the model is described as a Stuckelberg game between the dominant firm, the leader and the fringe firm as follower. The model allows to study the dynamics of market shares and the value function of both types of firms. We can show analytically and numerically by employing a dynamic programming algorithm that there are periodic as well as converging solutions feasible under certain parameter constellations. The model thus predicts that considerable fluctuations of market shares as well as the asset prices of firms might arise in industries with heterogenous firms.

Appendix

We here consider the dynamic optimization problem of the competitive fringe firm explicitly.
We can express the net cash flow of the fringe firm as follows

$$V = \int_0^\infty \{(P_t - c_f)x_t(1 - s_{ft})\}e^{-rt}dt \qquad (A1)$$

where $r > 0$ is the discount rate of the fringe firm which is assumed to be a positive constant. It may not be necessarily the same as the discount rate of the dominant firm, ρ. The dynamic constraint of the fringe firm is

$$\dot{x}_t = (P_t - c_f)x_t s_{ft} m, \quad x_0 > 0. \qquad (A2)$$

Here it is assumed that the source of the finance of the investment of the fringe firm is restricted to internal finance and there is no adjustment cost of investment for the fringe firm. The optimal policy of the fringe firm is given by

$$\max_{s_{ft}\in[0,\overline{s}_f]} \int_0^\infty \{(P_t - c_f)x_t(1 - s_{ft})\}e^{-rt}dt \qquad (A3)$$

subject to

$$\dot{x}_t = (P_t - c_f)x_t s_{ft} m, \quad x_0 > 0. \qquad (A4)$$

The movement of P_t is given exogenously to the fringe firm, because P_t is determined by the dominant firm. The only control variable of the fringe firm is the rate of internal retention (s_f), and $\overline{s}_f$ is the institutionally given upper bound of s_f (it is assumed that $0 < \overline{s}_f < 1$).

The current value Hamiltonian of this problem becomes

$$\begin{aligned} H_{ft} &= (P_t - c_f)x_t(1 - s_{ft}) + \lambda_{ft}(P_t - c_f)x_t s_{ft} m \\ &= (P_t - c_f)x_t\{(1 - s_{ft}) + \lambda_{ft} s_{ft} m\} \\ &= (P_t - c_f)x_t\{(m\lambda_{ft} - 1)s_{ft} + 1\} \qquad (A5) \end{aligned}$$

where λ_{ft} is the co-state variable and a set of the optimal conditions for the fringe firm is given as follows

$$\max_{s_{ft} \in [0, \overline{s}_f]} H_{ft} \ \text{ for all } \ t \geq 0. \tag{A6}$$

$$\dot{\lambda}_{ft} = -\frac{\partial H_{ft}}{\partial x_t} + r\lambda_{ft} \ \text{ for all } \ t \geq 0. \tag{A7}$$

$$\lim_{t \to \infty} \lambda_{ft} e^{-rt} = 0. \tag{A8}$$

Henceforth, we shall only consider the case of $P_t - c_f > 0$ for all $t \geq 0$. In this case, we have

$$\frac{\partial H_{ft}}{\partial s_{ft}} = (P_t - c_f) x_t (m\lambda_{ft} - 1) \begin{cases} > 0 & \text{if } m\lambda_{ft} > 1 \\ = 0 & \text{if } m\lambda_{ft} = 1 \\ < 0 & \text{if } m\lambda_{ft} < 1 \end{cases} \tag{A9}$$

Therefore, we obtain

$$s_{ft} = \begin{cases} \overline{s}_f & \text{if } m\lambda_{ft} > 1 \\ 0 & \text{if } m\lambda_{ft} < 1 \end{cases} \tag{A10}$$

and all of $s_{ft} \in [0, \overline{s}_f]$ are indifferent if $m\lambda_{ft} = 1$.
Next, it follows from eq. (A7) that

$$\begin{aligned} \dot{\lambda}_{ft} \quad &= \quad -(P_t - c_f)(1 - s_{ft}) \\ &+ \quad \{r - (P_t - c_f) s_{ft} m\} \lambda_{ft} \\ &\text{for all } \ t \geq 0. \end{aligned} \tag{A11}$$

Now, let us define $M_1(t)$ and $M_2(t)$ as follows.

$$M_1(t) \equiv m(1 - \overline{s}_f) \int_t^{\infty} (P_\tau - c_f) e^{-r(\tau - t)} d\tau \tag{A12}$$

$$M_2(t) \equiv m \int_t^{\infty} (P_\tau - c_f) e^{\{(P_\tau - c_f)\overline{s}_f m - r\}(\tau - t)} d\tau \qquad (A13)$$

It is obvious that $0 < M_1(t) < M_2(t)$ if $P_\tau - c_f > 0$ for all $\tau \geq 0$. We can prove the following proposition under the assumption that $P_\tau - c_f > 0$ for all $\tau \geq 0$.

Proposition A1

(i) Suppose that $M_1(t) > 1$ for all $t \in [t_0, t_1]$ where $0 \leq t_0 < t_1$. Then, we have $s_{ft} = \overline{s}_f$ for all $t \in [t_0, t_1]$ at the optimal path of the fringe firm.

(ii)Suppose that $M_2(t) < 1$ for all $t \in [t_2, t_3]$ where $0 \leq t_2, < t_3$. Then we have $s_{ft} = 0$ for all $t \in [t_2, t_3]$ at the optimal path of the fringe firm.

(Proof)

We can rewrite eq. (A11) as

$$\dot{\lambda}_{ft} + (g_{ft} - r)\lambda_{ft} = -(P_t - c_f)(1 - s_{ft}) \qquad (A14)$$

where

$$g_{ft} \equiv (P_t - c_f) s_{ft} m \geq 0. \qquad (A15)$$

Eq. (14) is equivalent to

$$\frac{d}{dt}(\lambda_{ft} e^{\int_0^t \{g_{fv} - r\} dv}) = -(P_t - c_f)(1 - s_{ft}) e^{\int_0^t \{g_{fv} - r\} dv} \qquad (A16)$$

Integrating this equation, we have

$$\lim_{\tau \to \infty} (\lambda_{f\tau} e^{\int_0^\tau \{g_{fv} - r\} dv}) \; - \; \lambda_{ft} e^{\int_0^t \{f_{fv} - r\} dv}) = -\int_0^{\infty} (P_\tau - c_f)(1 - s_{ft}) e^{\int_0^\tau \{g_{fv} - r\} dv} d\tau \qquad (A17)$$

The costate variable λ_{ft} must be positive, since λ_{ft} is the shadow price of x_t, and the increase of x_t contributes to the increase V (see Chiang (1992)). In this case, it follows from eq. (A11) that

$$\begin{aligned} g_{\lambda ft} \equiv \frac{\dot{\lambda}_{ft}}{\lambda_{ft}} &= -\frac{(P_t - c_f)(1 - s_{ft})}{\lambda_{ft}} + r - g_{ft} \\ &< r - g_{ft} \end{aligned} \qquad (A18)$$

for all $s_{ft} \in [0, \overline{s}_f]$. Therefore, we have

$$\begin{aligned} &\lim_{\tau \to \infty} (\lambda_{f\tau} e^{\int_0^\tau \{g_{fv} - r\} dv}) \\ &= \lim_{\tau \to \infty} (\lambda_{f0} e^{\int_0^\tau \{g_{\lambda fv} + g_{fv} - r\} dv}) = 0 \end{aligned} \qquad (A19)$$

It follows from equations (A17) and (A19) that

$$\lambda_{ft} = \int_0^\infty (P_\tau - c_f)(1 - s_{f\tau}) e^{\int_t^\tau \{g_{fv} - r\} dv} d\tau \qquad (A20)$$

It is clear from (A12), (A13) amd (A20) that

$$M_1(t) \leq m\lambda_{ft} \leq M_2(t) \qquad (A21)$$

for all $s_{ft} \in [0, \overline{s}_f]$, because

$$0 \leq g_{fv} = (P_v - c_f) s_{fv} m \leq (P_v - c_f) \overline{s}_f m \qquad (A22)$$

and $P_\tau - c_f > 0$ by assumption. From the inequalities (A21) we have $m\lambda_{ft} > 1$ whenever $M_1(t) < 1$, and we have $m\lambda_{ft} < 1$ whenever $M_2(t) < 1$. From these results and eq. (A10) we obtain the results of **Proposition A1**. (Q.E.D.)

Proposition A1 characterizes a typical bang-bang solution in which the opposite types of corner solutions switch discontinuously. The following result is a simple corollary of **Proposition A1**.

Corollary of Proposition A1

(i) We have $s_{ft} = \overline{s}_f$ for all $t \geq 0$ at the optimal path of the fringe firm if $M_1(t) > 1$ for all $t \geq 0$.
(ii) We have $s_{ft} = 0$ for all $t \geq 0$ at the optimal path of the fringe firm if $M_2(t) < 1$ for all $t \geq 0$.

Above we implicitly assumed the case of $M_1(t) > 1$ for all $t \geq 0$. It is more likely that $M_1(t) > 1$ is satisfied the greater are $(P_\tau - c_f)$ and m and the smaller are r and $\overline{s}_f$. In other words, if the profitability of the fringe firm is relatively high compared with the discount rate of the fringe firm and the institutionally given upper limit of the retention rate is relatively small, the financially constrained fringe firm will try to accumulate the capital up to the limit which is given by the limit of its internal finance, This conclusion is quite reasonable from the economic point of view.

Moreover, for the special case of $P_\tau = \overline{P}$ for all $\tau \geq 0$, the conditions $M_1(t) > 1$ is reduced to the following simple condition.

$$m(1 - \overline{s}_f)(\overline{P} - c_f) > r \qquad (A23)$$

Remark A1
For the inequality (A18) we have $g_{\lambda ft} < r$ so that we obtain

$$\lim_{t\to\infty} \lambda_{ft} e^{-rt} = \lim_{t\to\infty} \lambda_{f0} e^{\int_0^t \{g_{\lambda f\tau} - r\} d\tau} = 0 \qquad (A24)$$

In other words, the transversality condition (A8) is in fact satisfied in this case.

Remark A2
Substituting the relationship

$$x_\tau = x_t e^{\int_t^\tau g_{fv} dv} \quad \text{for } v \geq t \qquad (A25)$$

into eq. (A20), we have the following expression.

$$\begin{aligned} \lambda_{ft} x_t &= \int_0^\infty (P_\tau - c_f)(1 - s_{f\tau}) x_\tau e^{-r(\tau - t)} d\tau \\ &\equiv V_t \qquad (A26) \end{aligned}$$

where V_t is the 'value' of the fringe firm at the period t. Therefore, we have

$$m\lambda_{ft} = \frac{V_t}{(x_t/m)} = \frac{V_t}{K_{ft}} \equiv q_{ft} \qquad (A27)$$

where $V_t/K_{ft} \equiv q_{ft}$ is the 'Tobin's q' of the fringe firm. Hence, we can interpret eq. (A10) in terms of the Tobin's q theory even of it is assumed that the fringe firm has no free access to the capital market. Namely, we have the relationship

$$s_{ft} = \begin{cases} \overline{s}_f & \text{if } q_{ft} > 1, \\ 0 & \text{if } q_{ft} < 1. \end{cases} \qquad (A28)$$

By applying essentially the same reasoning we can show that the above co-state variable λ_t in fact becomes to be Tobin's q of the dominant firm (see Asada (1999).

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