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**History Dependence and Hysteresis Effects in an
Investment Model with Adjustment Costs**

by

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Abstract

This paper, following the line of research by Feichtinger, Hartl, Kort and Wirl (2001), henceforth FHKW (2001), studies history dependence and hysteresis effects of an investment model with relative adjustment cost. As in FHKW (2001) we study the existence of equilibria in concave and non-concave domains and the local stability properties of the equilibria. In contrast to FHKW (2001) by using the Hamiltonian-Jacobi-Bellman (HJB) equation we can study the global dynamics and thresholds which separate different domains of attraction. A numerical procedure derived from the HJB equation permits to locate those thresholds and to explore the global dynamics. The important implication of our paper is that in the standard investment model of the firm there will be appear history dependence and hysteresis effects if relative adjustment costs are admitted.

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1 Introduction

Multiple equilibria have been found in many dynamic economic models. Multiple equilibria give rise to history dependence and hysteresis effects. The property of history dependence comes about if solutions of dynamic systems can converge, depending on the initial conditions, toward distinct attractors. The existence of two or more stable steady states implies the existence of at least one unstable steady state. Moreover, the existence of optimal trajectories towards the one or the other attractor implies the existence of an optimal set of points, called thresholds or Skiba-points, at which one is indifferent between converging toward the one or the other stable steady state.

One of the simplest models which leads to multiple equilibria is based on a convex-concave production function. The article by Skiba (1978) is a seminal paper on multiple equilibria in the literature on economic development. He uses a convex-concave production function which possesses increasing returns to scale at an early stage of economic development and diminishing returns at a later stage. This gives a threshold, or a Skiba point, in a one state variable dynamic model.

In a recent survey paper by Deissenberg, Feichtinger, Semmler and Wirl (2001) it is shown that in many areas of economics dynamic optimization problems with multiple equilibria arise. They also pointed out the less known facts that multiple equilibria and history dependence are possible in the concave domain and the unstable steady states are not necessarily optimal and do not always coincide with the thresholds.

The current paper, following the work by Feichtinger, Hartl, Kort and Wirl (2001), studies history dependence and hysteresis effects in an investment model with adjustment costs. Investment models with absolute adjustment costs have been studied in the work by Eisner and Stroz (1963), Lucas (1967), Gould (1968) and Mortenson (1973). Models with relative adjustment costs can be found in Uzawa (1968, 1969), Hayashi (1982) and D'Autume and Michel (1985). The latter type of models frequently imply the existence of multiple equilibria and history dependence.

We also study a dynamic decision problem of investment by a firm. We follow Feichtinger, Hartl, Kort and Wirl (2001), henceforth FHKW (2001), who have presented a dynamic investment model with relative adjustment costs and solve this model by using Pontryagin's maximum principle and the associated Hamiltonian. They, however, cannot exactly locate the threshold using their method. We solve the model by using the Hamiltonian-Jacobi-Bellman (HJB)-equation, can locate the threshold and study the global dynamics.

The remainder of the paper is organized as follows. Section 2 presents

the FHKW (2001) model. Section 3 shows the existence of concave-non-concave domains of the model. Section 4 studies the global dynamics of the model employing the HJB-equation. Thresholds are obtained by numerical methods. Section 5 concludes the paper.

2 Relative Adjustment Costs and Multiple Equilibria

FHKW (2001) study a dynamic investment model and investigate a number of interesting properties when agents face relative adjustment costs. There are not only multiple equilibria but also equilibria within the concave domain. Their model can be summarized as follows:

$$\max_{\{u(t)\}} \int_0^{\infty} \exp(-rt)[U(x(t)) - C(u(t)/x(t))]dt \quad (1)$$

s.t.

$$\dot{x}(t) = u(t) - \delta x(t), \quad x(0) = x_0 \quad (2)$$

The capital stock x generates a concave pay-off, U , and C denotes the adjustment costs depending on the investment to capital stock ratio. This cost function is not jointly convex, although it is convex in u and x separately. This can be verified by the negative of the determinant of the Hessian matrix X .

$$X = \begin{bmatrix} C''(\frac{u}{x})\frac{u^2}{x^4} + C'(\frac{u}{x})\frac{2xu}{x^4} & -\frac{1}{x^2}[C''(\frac{u}{x})\frac{u}{x} + C'(\frac{u}{x})] \\ -\frac{1}{x^2}[C''(\frac{u}{x})\frac{u}{x} + C'(\frac{u}{x})] & C''(\frac{u}{x})\frac{1}{x^2} \end{bmatrix} \quad (3)$$

$$\det |X| = -\left(\frac{C'(u/x)}{x^2}\right)^2 < 0 \quad (4)$$

As a consequence, the integrand function ($U - C$) and the associated Hamiltonian need not be concave in particular around small steady states. The possible existence of both concave and non-concave domains leads to multiple equilibria. Yet one of their interesting findings is that neither the existence of multiple equilibria nor the potential instability and the associated threshold property require non-concavity.

The current value Hamiltonian H , using λ to denote the co-state variable of x , is

$$H = U(x) - C\left(\frac{u}{x}\right) + \lambda(u - \delta x). \quad (5)$$

The first order conditions for interior solutions are

$$H_u = -\frac{C'}{x} + \lambda = 0 \quad (6)$$

$$\dot{\lambda} = (r + \delta) - U' - C' \frac{u}{x^2}. \quad (7)$$

The second order necessary condition for optimality, the so-called Legendre-Clebsch condition, is satisfied if

$$H_{uu} = -\frac{C''}{x^2} = -C_{uu} < 0. \quad (8)$$

The necessary condition is sufficient if the Hamiltonian is concave in (u, x) (the Mangasarian sufficiency theorem)¹ or the maximized Hamiltonian is concave in X (the Arrow sufficiency theorem). Since the analysis is restricted to the necessary conditions, a slight vagueness remains on the optimality of the paths characterized in the non-concave domain.

From the maximum principle (6),

$$u^* = u(x, \lambda), \quad (9)$$

where, introducing $\sigma = \frac{(u/x)C''}{C'}$, the elasticity of marginal costs, gives

$$u_x = \frac{u}{x} \frac{(1 + \sigma)}{\sigma} \quad (10)$$

$$u_\lambda = \frac{x^2}{C''(u/x)}. \quad (11)$$

At a steady state, $\dot{x} = u - \delta x = 0$ or $u/x = \delta$, thus (10) becomes

$$u_x = \frac{\delta(1 + \sigma)}{\sigma}. \quad (12)$$

Substituting (9) into (2) and (7) yields the canonical equations in (x, λ) :

¹Consider the following problem:

$$\max V = \int_0^\infty F(t, u, x) dt$$

s.t.

$$\dot{x} = f(t, u, x) \quad x(0) = x_0.$$

In this version sufficiency holds if both the F and f functions are differentiable and concave in the variables (u, x) jointly, and for the optimal solution it is true that $\lambda(t) \geq 0$ for all t if f is nonlinear in u or in x .

$$\dot{x} = u(x, \lambda) - \delta x \quad (13)$$

$$\dot{\lambda} = (r + \delta)\lambda - U'(x) - C' \left(\frac{u(x, \lambda)}{x} \right) \frac{u(x, \lambda)}{x^2}. \quad (14)$$

The Jacobian J , associated with the canonical equations (13) and (14) evaluated at a steady state, and its determinant are given by

$$J = \begin{bmatrix} \frac{\delta}{\sigma} & \frac{x^2}{C''} \\ -\frac{C''}{x^2} - U'' & r - \frac{\delta}{\sigma} \end{bmatrix} \quad (15)$$

$$\det |J| = \frac{r\delta}{\sigma} + \frac{x^2 U''}{C''}. \quad (16)$$

The sign of $\det |J|$ is ambiguous. The last term of $\det |J|$ is definitely negative due to the concavity of U . Therefore, no depreciation, $\delta = 0$, guarantees a stable steady state. For higher r and δ or lower σ , the $\det |J|$ increases and thus the possibility of instability rises.

Taking this model by FHKW (2001) we solve it by using both the maximum principle and the HJB-equation in order to study thresholds and history dependence.

$$\max_{\{u(t)\}} \int_0^\infty \exp(-rt) [U(x(t)) - C(u(t)/x(t))] dt \quad (17)$$

s.t.

$$\dot{x}(t) = u(t) - \delta x(t), \quad x(0) = x_0 \quad (18)$$

with the following specifications:

$$C(z) = \frac{1}{2}cz^2 = \frac{1}{2}c \left(\frac{u}{x} \right)^2 \quad (19)$$

$$U(x) = x - \frac{1}{2}x^2. \quad (20)$$

where we employ a linear-quadratic pay-off function and a quadratic adjustment costs function.

The current value Hamiltonian H is

$$H = x - \frac{1}{2}x^2 - \frac{1}{2}c \left(\frac{u}{x} \right)^2 + \lambda(u - \delta x). \quad (21)$$

The first order conditions for interior solutions are

$$H_u = -c \left(\frac{u}{x} \right) \frac{1}{x} + \lambda = 0 \quad (22)$$

$$\dot{\lambda} = (r + \delta)\lambda - (1 - x) - c\left(\frac{u}{x}\right)^2 \frac{1}{x}. \quad (23)$$

From the maximum principle (22) we can get the optimal investment policy:

$$u^* = \frac{\lambda x^2}{c}. \quad (24)$$

Substituting the optimal policy (24) into (23) gives us

$$\dot{\lambda} = (r + \delta)\lambda - (1 - x) - \frac{\lambda^2 x}{c}. \quad (25)$$

Also substituting (24) into the state equation (15) gives us

$$\dot{x} = x\left(\frac{\lambda x}{c} - \delta\right). \quad (26)$$

Consequently the system is described by the canonical equations (25) and (26).

By setting $\dot{x} = 0$, and $\dot{\lambda} = 0$, we find that this system has three steady states:

$$(x_\infty, \lambda_\infty) = \begin{cases} \left(0, \frac{1}{r+\delta}\right) \\ \left(\frac{1-\sqrt{D}}{2}, \frac{1-\sqrt{D}}{2r}\right) \\ \left(\frac{1+\sqrt{D}}{2}, \frac{1-\sqrt{D}}{2r}\right). \end{cases} \quad (27)$$

Here we assume $D \equiv 1 - 4cr\delta \geq 0$, otherwise only the first and trivial steady state, $x_\infty = 0$, exists.

For the study of phase diagram we need to note that we have two $\dot{x} = 0$ isoclines and three $\dot{\lambda} = 0$ isoclines in the (x, λ) plane:

$$\dot{x} = 0, \text{ if either } \begin{cases} x = 0 \\ \lambda = \frac{c\delta}{x} \end{cases} \text{ or} \quad (28)$$

$$\dot{\lambda} = 0, \text{ if } \lambda = \begin{cases} \frac{1}{r+\delta} & \text{for } x = 0 \\ \frac{c(r+\delta) \pm \sqrt{c^2(r+\delta)^2 + 4cx(x-1)}}{2x} & \text{for } x > 0, \end{cases} \quad x \in [0, 1]. \quad (29)$$

The domain of the $\dot{\lambda} = 0$ isocline is restricted to the states where $c(r + \delta)^2 > 4x(1 - x)$ is satisfied.

Next, we undertake a stability analysis of system (25)-(26). The Jacobian of the canonical equations system is given by

$$J = \begin{bmatrix} -\delta + \frac{2\lambda x}{c} & \frac{x^2}{c} \\ 1 - \frac{\lambda^2 c}{c} & r + \delta - \frac{2\lambda x}{c} \end{bmatrix} \quad (30)$$

Because $\lambda = c\delta/x$ holds for the interior steady states, $x > 0$, J will be reduced to

$$J = \begin{bmatrix} \delta & \frac{x^2}{c} \\ \frac{x^2 - c\delta^2}{x^2} & r - \delta \end{bmatrix} \quad (31)$$

while for the boundary steady state, $x = 0$, $\lambda = 1/(r + \delta)$ holds, and J will be reduced to

$$J = \begin{bmatrix} -\delta & 0 \\ 1 - \frac{1}{(r+\delta)^2 c} & r + \delta \end{bmatrix}. \quad (32)$$

Thus the determinant of the Jacobian at the three steady state is

$$\det | J | = \begin{cases} -\delta(r + \delta) < 0 \\ \delta r - \frac{x^2}{c} = \begin{cases} \frac{\sqrt{D}-D}{2c} > 0 & \text{for } x = 0 \\ -\frac{\sqrt{D}+D}{2c} < 0 & \text{for } x > 0 \end{cases} \end{cases} \quad (33)$$

Since $\det | J | < 0$ for the smallest ($x = 0$) and the largest steady states, these two steady states are locally stable, while the middle steady state is unstable since $\det | J | > 0$.

3 Concave and Non-Concave Domains

Another interest of the paper of the FHKW (2001) is to study whether the equilibria fall into concave or non-concave domains. We are especially interested in the location of the middle unstable steady state. Following FHKW (2001) we can reveal concavity and non-concavity by checking the determinant of the Hessian of the Hamiltonian at the steady states.

The Hessian matrix A of the Hamiltonian at the two positive steady states and its determinant are

$$A = \begin{bmatrix} -\frac{c}{x^2} & \frac{2cu}{x^3} \\ \frac{2cu}{x^3} & -\left(1 + \frac{3cu^2}{x^4}\right) \end{bmatrix} \quad (34)$$

$$\det | A | = \frac{c}{x^4} \left[x^2 - c \left(\frac{u}{x} \right)^2 \right]. \quad (35)$$

Since $\dot{x} = u - \delta x = 0$ or $u/x = \delta$ at the two positive steady states,

$$\det | A | = \frac{c}{x^4} [x^2 - c\delta^2]. \quad (36)$$

The sign of $\det | A |$ will be known when we insert the steady state values of x in (36):

$$\det | A | = \frac{8c[1 - 2c\delta(r + \delta) \pm \sqrt{D}]}{(1 \pm \sqrt{D})^4} \quad (37)$$

where the positive sign before the root corresponds to the largest steady state and the negative sign to the middle steady state. From (37), the Hamiltonian is concave for any $x > \delta\sqrt{c}$, even if the Jacobian is positive; $x < \sqrt{r\delta c}$. This suggests that examples that couple instability and concavity are easy to construct if the necessary requirement, $r > \delta > 0$, is met.

For a low depreciation rate $r > \delta$, by decreasing the value of c , the middle steady state moves from the non-concave domain to the concave domain. Interestingly, the determinant of the Hessian matrix of the Hamiltonian vanishes at that point where the domain of the co-state isocline begins to be restricted. On the other hand, for a high depreciation rate, the middle steady state is unstable within the non-concave domain and the largest stable steady state is within the concave domain. It is also noteworthy that the property of the unstable steady state within the non-concave domain can be an unstable node.

Let us concentrate more properly on the unstable steady state. Each steady state, including the unstable one in the concave domain must be a node, while the unstable steady state in the non-concave domain can be either a node or a focus. These results show that it is not necessary for an unstable steady state to be in the non-concave domain, and moreover, the unstable steady state in the non-concave domain can be a node. This outcome is, as already FHKW (2001) state, surprising because an extensive literature on multiple equilibria creates the opposite impression namely that an instability coupled with a non-concavity implies an unstable focus.

We now check that the unstable steady state in the concave domain must be a node. Concavity requires that the determinant of the Hessian is positive:

$$\det | A | = \frac{8c[1 - 2c\delta(r + \delta) - \sqrt{D}]}{(1 \pm \sqrt{D})^4} > 0, \quad (38)$$

which is equivalent to

$$1 - 2c\delta(r + \delta) - \sqrt{D} > 0.$$

Rearranging the latter gives the following condition:

$$c\delta(r - \delta) > \frac{1}{2}(1 - \sqrt{D})\sqrt{D}. \quad (39)$$

The stability property depends on the eigenvalues at the unstable steady state. Recall that the Jacobian of the system evaluated at the interior steady state is

$$J = \begin{bmatrix} \delta & \frac{x^2}{c} \\ \frac{x^2 - c\delta^2}{x^2} & r - \delta \end{bmatrix} \quad (40)$$

and thus the determinant of the Jacobian at the unstable steady state is

$$\det |J| = \frac{\sqrt{D} - D}{2c} > 0. \quad (41)$$

The characteristic equation of the Jacobian is

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \quad (42)$$

where

$$\begin{aligned} -(a_{11} + a_{22}) &= -(\delta + r - \delta) \\ &= -r \end{aligned}$$

and

$$(a_{11}a_{22} - a_{12}a_{21}) = \frac{\sqrt{D} - D}{2c}.$$

The characteristic equation, $\lambda^2 - r\lambda + (\sqrt{D} - D)/2c$, is satisfied for two values of λ :

$$\lambda = \frac{1}{2} \left[r \pm \sqrt{\frac{cr^2 + 2(D - \sqrt{D})}{c}} \right]. \quad (43)$$

The supposition of a node requires that two eigenvalues at the unstable steady state are real, which is equivalent to $cr^2 + 2(D - \sqrt{D}) > 0$ or

$$\frac{1}{2}(1 - \sqrt{D})\sqrt{D} < \frac{1}{4}cr^2. \quad (44)$$

On the other hand, the supposition of a focus requires two eigenvalues are complex conjugate, which is equivalent to

$$\frac{1}{2}(1 - \sqrt{D})\sqrt{D} > \frac{1}{4}cr^2. \quad (45)$$

In addition, it is obvious that²

$$c\delta(r - \delta) < \frac{1}{4}cr^2. \quad (46)$$

Consequently, it should be clear that the only possible situation under the condition where (39) holds, that is where the unstable steady state is in the concave domain, is

$$\frac{1}{2}(1 - \sqrt{D})\sqrt{D} < c\delta(r - \delta) < \frac{1}{4}cr^2. \quad (47)$$

It implies that the unstable steady state in the concave domain must be node. In other words, an unstable focus is impossible in the concave domain.

These results are illustrated by using a numerical example. Let $c = \frac{3}{2}$, $r = 1$, $\delta = 0.1$ with the steady states for x : 0, 0.184, 0.816. At the middle unstable steady state $x=0.184$ we have

$$\frac{1}{2}(1 - \sqrt{D})\sqrt{D}[= 0.116] < c\delta(r - \delta)[= 0.135] < \frac{1}{4}cr^2[= 0.375]$$

Therefore the unstable steady state in the concave domain must be a node.

In the same way, we can check that the unstable steady state in the non-concave domain can be not only a focus but also a node. Non-concavity requires that the determinant of the Hessian is negative.

$$\det | A | = \frac{8c[1 - 2c\delta(r + \delta) - \sqrt{D}]}{(1 \pm \sqrt{D})^4} < 0, \quad (48)$$

which is equivalent to

$$1 - 2c\delta(r + \delta) - \sqrt{D} < 0$$

Rearranging this gives the following condition:

$$c\delta(r - \delta) < \frac{1}{2}(1 - \sqrt{D})\sqrt{D}. \quad (49)$$

²Note that $\frac{1}{4}cr^2 - c\delta(r - \delta) = \frac{1}{4}c(r - 2\delta)^2 > 0$

From (44), (45) and (46), under the condition where (49) holds, that is where the unstable steady state is in the non-concave domain, there are two possible situations:

$$c\delta(r - \delta) < \frac{1}{2}(1 - \sqrt{D})\sqrt{D} < \frac{1}{4}cr^2 \quad (50)$$

$$c\delta(r - \delta) < \frac{1}{4}cr^2 < \frac{1}{2}(1 - \sqrt{D})\sqrt{D}. \quad (51)$$

(50) implies that the unstable steady state in the non-concave domain can be a node, while (51) implies that it can also be a focus. Those outcomes depend on the parameters of the model. This can be shown with the following parameters: $c = 1$, $r = \frac{1}{4}$, $\delta = \frac{1}{2}$. This gives steady states for x : 0, 0.146, 0.854, and the unstable one at $x = 0.146$.

We thus have

$$c\delta(r - \delta)[= -0.125] < \frac{1}{4}cr^2[= 0.016] < \frac{1}{2}(1 - \sqrt{D})\sqrt{D}[= 0.104]$$

Therefore this example shows the unstable steady state in the non-concave domain must be a focus.

4 The Study of the Global Dynamics

The numerical computation of the value function and the global dynamics follows Semmler (1999) and Semmler and Sieveking (2000). There it is also shown how the thresholds, or Skiba points, can be computed by using the HJB-equation.

Employing again the FHKW (2001) investment model with relative adjustment cost

$$\max \int_0^{\infty} e^{-rt} f(x, u) dt \quad (52)$$

s.t.

$$\dot{x} = g(x) = u(x) - \delta x(t) \quad (53)$$

we can define the optimal value function $J(t_0, x_0)$ as the maximum value that can be obtained starting at time t_0 at state x_0 :

$$\begin{aligned} J(t_0, x_0) &= \max \int_0^{\infty} e^{-rt} f(x, u) dt \\ &= e^{-rt_0} \max \int_0^{\infty} e^{-r(t-t_0)} f(x, u) dt \end{aligned} \quad (54)$$

The value of the integral on the right hand side depends on the initial state, but is also dependent on the initial time, i.e. it depends on elapsed time.

Now, let us define

$$V(x_0) = \max \int_0^{\infty} e^{-r(t-t_0)} f(x, u) dt. \quad (55)$$

Then

$$J(t, x) = e^{-rt} V(x) \quad (56)$$

$$J_t = -r e^{-rt} V(x) \quad (57)$$

$$J_x = e^{-rt} V'(x). \quad (58)$$

The HJB-equation is

$$-J_t(t, x) = \max_u [e^{-rt} f(t, x, u) + J_x(t, x) g(x, u)]. \quad (59)$$

Substituting (57) and (58) into (59) and multiplying through by e^{rt} yields the basic ordinary differential equation:

$$rV(x) = \max_u [f(x, u) + V'(x)g(x, u)] \quad (60)$$

We employ again the model by FHKW (2001) with the following specific functions:

$$f(x, u) = U(x) - C\left(\frac{u}{x}\right) = x - \frac{1}{2}x^2 - \frac{1}{2}c\left(\frac{u}{x}\right)^2 \quad (61)$$

$$g(x, u) = u - \delta x \quad (62)$$

Substituting (61) and (62) into (60) gives

$$rV(x) = \max_u [U(x) - C\left(\frac{u}{x}\right) + V'(x)(u - \delta x)]. \quad (63)$$

Next we compute the candidates for steady state equilibria. If e is an equilibrium,

$$g(e, u) = u - \delta e = 0 \quad (64)$$

Then

$$rV(e) = U(e) - C(\delta) \quad (65)$$

$$V'(e) = \frac{U'(e)}{r + \delta} + \frac{C'(\delta)\delta}{(r + \delta)e} \quad (66)$$

The equilibrium e satisfies

$$rV(e) = \max_u [U(e) - C\left(\frac{u}{e}\right) + V'(e)(u - \delta e)], \quad (67)$$

and substituting (65) and (66) into (67) yields

$$U(e) - C(\delta) = \max_u \left[U(e) - C\left(\frac{u}{e}\right) + \left(\frac{U'(e)}{r + \delta} + \frac{C'(\delta)\delta}{(r + \delta)e} \right) (u - \delta e) \right]. \quad (68)$$

Then solving $\frac{d}{du}[\cdot] = 0$

$$-C\left(\frac{u}{e}\right) \frac{1}{e} + \frac{U'(e)}{r + \delta} + \frac{C'(\delta)\delta}{(r + \delta)e} = 0. \quad (69)$$

From the specific function (61),

$$U'(x) = 1 - x \quad (70)$$

$$C'\left(\frac{u}{x}\right) = c\left(\frac{u}{x}\right). \quad (71)$$

Then the equilibrium condition (69) becomes

$$-c\frac{u}{e}(r + \delta) + (1 - e)e + c\delta^2 = 0 \quad (72)$$

$$\text{or } u = \frac{(1 - e)e^2}{c(r + \delta)} + \frac{\delta^2 e}{r + \delta}.$$

Substituting this condition into the steady-state condition, we obtain three steady-state equilibria from

$$\begin{aligned} \dot{x} = u - \delta e &= \frac{(1 - e)e^2}{c(r + \delta)} + \frac{\delta^2 e}{r + \delta} - \delta e \\ &= e \left[\frac{(r - e)e}{c(r + \delta)} + \frac{\delta^2}{r + \delta} - \delta \right] \\ &= 0. \end{aligned} \quad (73)$$

It implies

$$\left[\frac{(1-e)e}{c(r+\delta)} + \frac{\delta^2}{r+\delta} - \delta \right] = 0, \text{ that is } e^2 - e + rc\delta = 0.$$

Thus, the optimal three steady states are

$$e = \begin{cases} 0 \\ \frac{1 \pm \sqrt{1-4rc\delta}}{2} = \frac{1 \pm \sqrt{D}}{2} \end{cases} \quad (74)$$

where we assume $D \equiv 1 - 4rc\delta \geq 0$.

Figure 1 shows the optimal steady states where both of the following conditions, are satisfied:

$$u = \frac{1-e}{rc} e^2 \quad (75)$$

$$u = \delta e \quad (76)$$

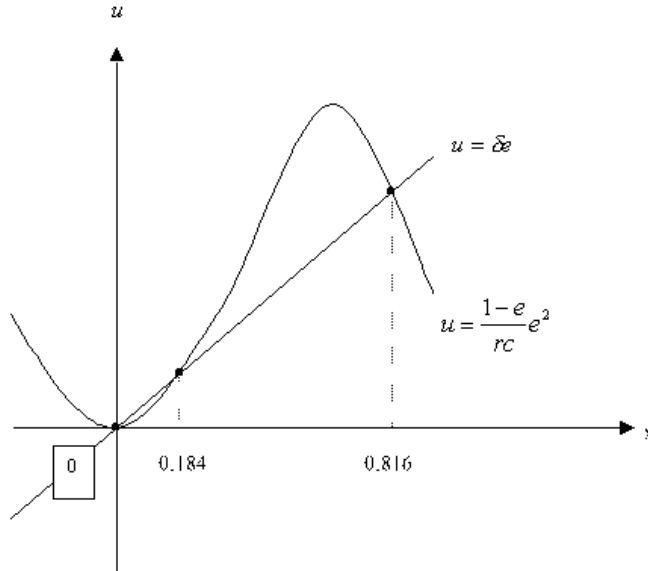


Figure 1: Example for parameter set $c=1.5$, $r=1$ and $\delta=0.1$

The second-order necessary condition requires

$$\frac{d^2}{du^2} \left[U(e) - C\left(\frac{u}{e}\right) + \frac{1}{r} U'(e)(u - \delta e) \right] < 0 \quad (77)$$

$$-C''\left(\frac{u}{e}\right)\frac{1}{e^2} < 0. \quad (78)$$

From the specific function (61) we have

$$C''\left(\frac{u}{x}\right) = c. \quad (79)$$

Thus, the condition (64) becomes

$$-\frac{c}{e^2} < 0. \quad (80)$$

This second-order necessary condition holds for the two positive optimal steady states.

The HJB-equation gives

$$\begin{aligned} rV(e) &= \max_u [f(e, u) + V'(e)g(e, u)] \\ &= \max_u \left[e - \frac{1}{2}e^2 - \frac{1}{2}c\left(\frac{u}{e}\right)^2 + V'(e)(u - \delta e) \right]. \end{aligned} \quad (81)$$

Solving $\frac{d}{du} \left[e - \frac{1}{2}e^2 - \frac{1}{2}c\left(\frac{u}{e}\right)^2 + V'(e)(u - \delta e) \right] = 0$ derives

$$\begin{aligned} -c\left(\frac{u}{e}\right)\frac{1}{e} + V'(e) &= 0 \quad \text{or} \\ u &= \frac{e^2}{c}V'(e). \end{aligned} \quad (82)$$

Substituting (82) into (81) gives

$$rV(e) = e - \frac{1}{2}e^2 + \frac{1}{2}\frac{e^2}{c}V'(e)^2 - \delta eV'(e) \quad (83)$$

therefore

$$V'(e)^2 - 2\frac{c\delta}{e}V'(e) + 2\frac{c}{e} - c - 2\frac{cr}{e^2}V(e) = 0.$$

Then we obtain an ordinary differential equation in V with the candidates of steady states as initial conditions:

$$V'(e) = \frac{c\delta}{e} \pm \sqrt{\left(\frac{c\delta}{e}\right)^2 - \left(2\frac{c}{e} - c - 2\frac{cr}{e^2}V(e)\right)}. \quad (84)$$

Using the following information for V :

$$V'(x) = \frac{c\delta}{x} - \sqrt{\left(\frac{c\delta}{x}\right)^2 - \left(2\frac{c}{x} - c - 2\frac{cr}{x^2}v(x)\right)} \quad \text{for } x \geq e \quad (85)$$

$$V'(x) = \frac{c\delta}{x} + \sqrt{\left(\frac{c\delta}{x}\right)^2 - \left(2\frac{c}{x} - c - 2\frac{cr}{x^2}V(x)\right)} \quad \text{for } x < e \quad (86)$$

$$V(e) = \frac{1}{r}\left[e - \frac{1}{2}e^2 - \frac{1}{2}c\delta^2\right], \quad (87)$$

and solving the ODE in V by the Euler method for each e as initial condition we can compute the global value function for the original problem:

$$V(x) = \text{Max}V. \quad (88)$$

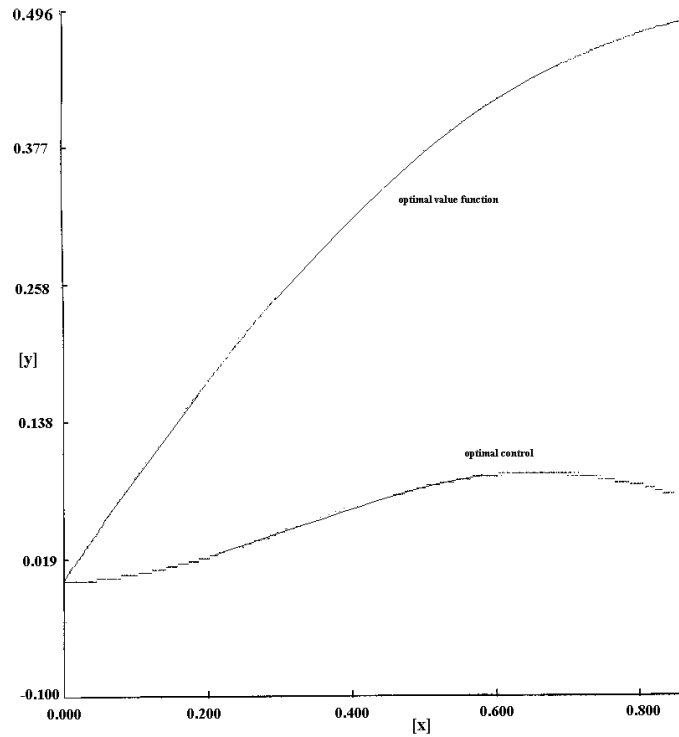


Figure 2: Example for parameter set $c=1.5$, $r=1$ and $\delta=0.1$

We can summarize the following results. In case of our revenue function and the quadratic adjustment costs, we get three steady states. One steady

state is unstable which gives rise to a threshold, in the present case where the unstable steady state falls in the concave domain and is a node the threshold coincides with the middle, unstable steady state. As above shown for a slow depreciation process, $r > \delta$, both positive (unstable and stable) steady states may fall into the concave domain. Yet, if this is a focus the unstable steady state and the threshold do not necessarily coincide. For a fast depreciation process, $r < \delta$, the unstable steady state is always in the non-concave domain.

Figure 2 shows the value function and the optimal control for the case of $r < \delta$. Figure 3 displays the kink in the value function indicating the threshold, the Skiba point, separating the different domains of attraction.

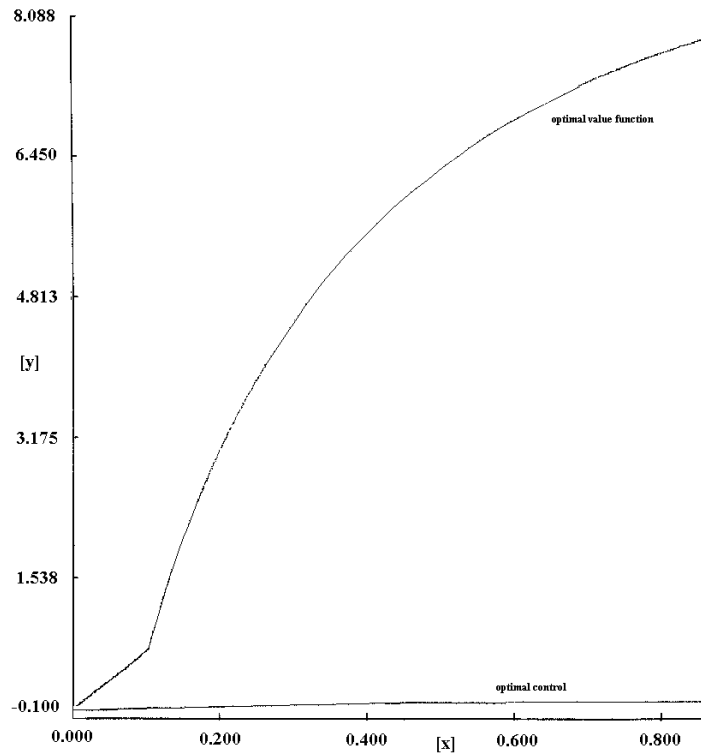


Figure 3: Example for parameter set $c=20$, $r=0.05$ and $\delta=0.1$

Moreover, figure 4 shows more clearly how the control variable in the vicinity of the middle unstable steady state jumps. Given this threshold the capital stock should be run down for initial conditions below the threshold. Above the threshold a high level of the state should be approached.

In the appendix, figure A1, the corresponding phase diagram to the value function and control variable of figure 2 are shown. For the parameter constellation of figure 2 the middle equilibrium is a node. Figures A2 and A3

of the appendix show the corresponding phase diagrams to figures 3 and 4 above. Figures 3 and 4 above and figures A2 and A3 in the appendix represent the case of a focus of the middle equilibrium. Figure A3 in the appendix shows the magnified region about the middle equilibrium. The value of the threshold, obtained from our numerical procedure, lies at $x = 0.105$ and thus it is located slightly to the left of the middle equilibrium.

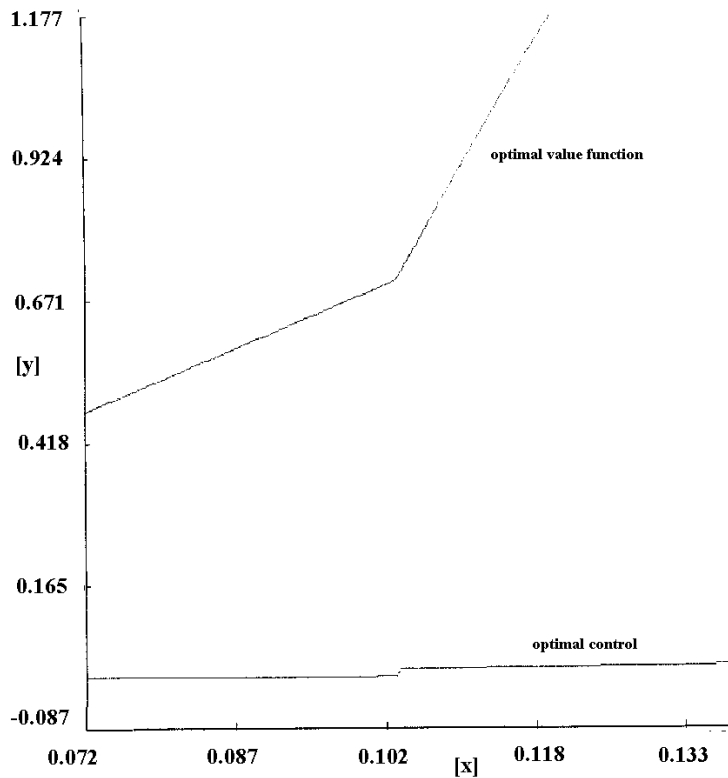


Figure 4: Example for parameter set $c=20$, $r=0.05$ and $\delta=0.1$

From a policy point of view, it is important to distinguish an unstable focus and unstable node, because in the case of a node as shown in Figure 2, investment is a continuous function of the capital stock level, while in the case of a focus, as in Figures 3 and 4, the policy function is always discontinuous.

5 Conclusions

In recent times numerous papers have been published that exhibit multiple equilibria. Multiple equilibria may lead to thresholds separating different domains of attraction, history dependence and hysteresis effects. Yet, detecting those phenomenon in a given model is cumbersome. Following Feichtinger, Hartl, Kort and Wirl (2001) in this paper we study an investment model with relative adjustment costs. This type of model implies the existence of multiple equilibria and history dependence. FHKW (2001) solve the model by using Pontryagin's maximum principle and the associated Hamiltonian. We solve the model by employing the Hamiltonian-Jacobi-Belman (HJB)-equation. By using the HJB equation we can analytically and numerically study the global dynamics, thresholds and history dependence. The numerical procedure that we are using is derived from the HJB equation. It permits to locate those thresholds and to explore history dependence of the model. The important implication of our paper is that in the standard investment model of the firm there will appear history dependence and hysteresis effects if relative adjustment costs are admitted.

6 Appendix: Phase Diagrams

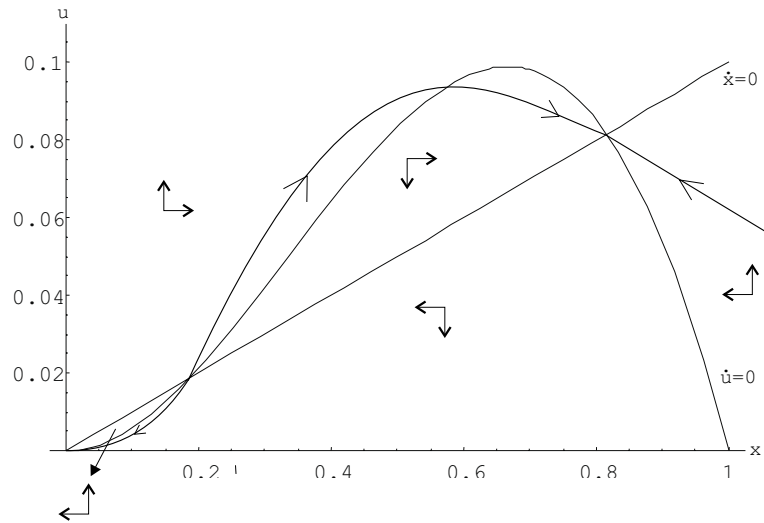


Figure A1:

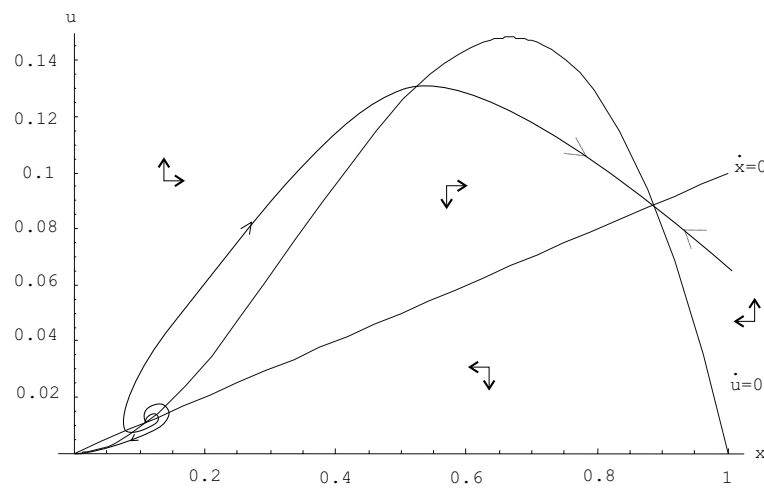


Figure A2:

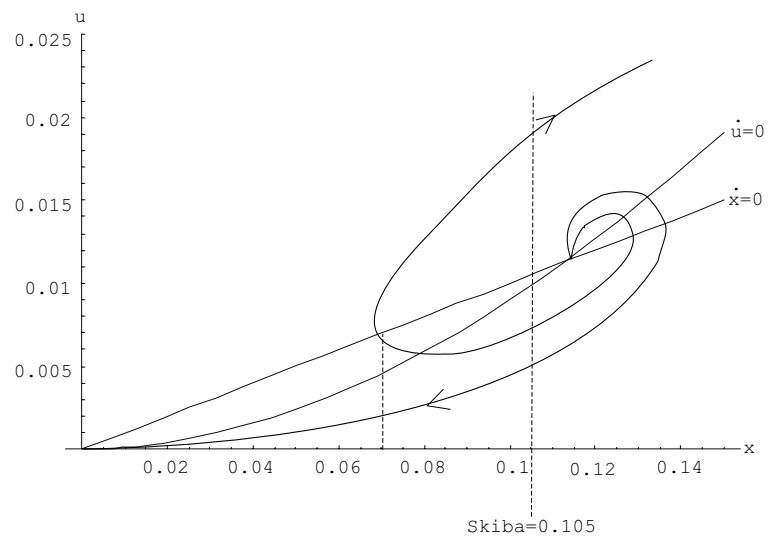


Figure A3:

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