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of a Stochastic Growth Model with
Asset Market Restrictions**

by

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Abstract

This paper estimates the parameters of a stochastic growth model with asset market and contrasts the model's moments with moments of the actual data. We solve the model through log-linearization along the line of Campbell (1994) and estimate the model without and with asset pricing restrictions. As asset pricing restrictions we employ the risk-free interest rate and the Sharpe-ratio. To estimate the parameters we employ, as in Semmler and Gong (1996a), a ML estimation. The estimation is conducted through the simulated annealing. We introduce a diagnostic procedure which is closely related to Watson (1993) and Diebold, Ohanian and Berkowitz (1995) to test whether the second moments of the actual macroeconomic time series data are matched by the model's time series. Several models are explored. The overall results are that sensible parameter estimates may be obtained when the actual and computed risk-free rate is included in the moments to be matched. The attempt, however, to include the Sharpe-ratio as restriction in the estimation does not produce sensible estimates. The paper thus shows, by employing statistical estimation techniques, that the baseline real business cycle (RBC) model is not likely to give correct predictions on asset market pricing when parameters are estimated from actual time series data.

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1 Introduction

The real business cycle model (RBC) has become one of the standard macroeconomic models. It tries to explain macroeconomic fluctuations as equilibrium reactions of a representative agent economy with complete markets. Many refinements have been introduced since the seminal papers by Kydland and Prescott (1982) and Hansen (1985) to improve the model's fit with the data. Usually a RBC model is calibrated rather than estimated by some rigorous statistical procedure. Moreover, the implications for asset prices are often ignored in the calibration process. This is surprising since asset prices contain valuable information about intertemporal decision making which is at the heart of the RBC methodology. This paper tries to estimate the parameters of a standard RBC model taking its asset pricing implications into account.

Modeling risk premia in models with production is much more challenging than in exchange economies. Most of the asset pricing literature has followed Lucas (1978) and Mehra and Prescott (1985) in computing asset prices in exchange economies. Production economies offer a much richer, and realistic environment. First, in exchange economies consumers are forced to consume their endowment. Consumers in production economies can save and hence transfer consumption between periods. Second, in exchange economies the aggregated consumption is usually used as a proxy for equity dividends. Empirically this is not a very sensible modeling choice. Since there is a capital stock in production economies a more realistic modeling of equity dividends is possible. We should stress that we do not attempt to solve the equity premium puzzle. Instead, we take a more macroeconomic view and focus on the impact of asset market restrictions on the estimation of real business cycle models.

Christiano and Eichenbaum (1992) use a generalized methods of moments (GMM) procedure to estimate the RBC parameters. Their moment restrictions only concern the real variables of the model. Semmler and Gong (1996a) estimate the model using Maximum Likelihood method. The purpose of this paper is to extend these papers by taking restrictions on asset prices implied by the RBC model into account when estimating the parameters of the model. We introduce the asset pricing restrictions step-by-step to clearly demonstrate the effect of each new restriction. As will become clear, the more asset market restrictions are introduced, the more difficult it becomes

to estimate the model. First we estimate the model using only restriction of real variables as in Christiano and Eichenbaum (1992) and Semmler and Gong (1996a). We obtain similar estimates for important parameters like risk aversion, discount factor and depreciation. The first additional restriction is the risk-free interest rate. We match the observed 30-day T-bill rate to the one-period risk-free rate implied by the model.¹ We find that the estimates are fairly close to those obtained without the additional restriction suggesting that the model's prediction of the risk-free rate is broadly consistent with the data.

The second additional asset pricing restriction concerns the risk-return trade-off implied by the model as measured by the Sharpe-ratio, or the price of risk. This variable determines how much expected return agents require per unit of financial risk. Hansen and Jagannathan (1992) and Lettau and Uhlig (1999) show how the Sharpe-ratio can be used to evaluate the ability of different models to generate high risk premia (see also Sharpe (1964)). Introducing a Sharpe-ratio as moment restriction to the estimation procedure requires an iterative procedure in order to estimate the risk aversion parameter. We find that the Sharpe-ratio restrictions affects the estimation of the model drastically. In particular, the parameters become unbounded and hence the model can no longer be estimated. In other words, the model cannot fulfill moment restrictions concerning real variables and the Sharpe-ratio simultaneously. The problem is that matching the Sharpe-ratio requires high risk aversion which on the other hand is incompatible with the observed variability of consumption. This tension which is at the heart of the model makes it impossible to estimate it. We experiment with various versions of the model, e.g. fixing risk aversion at a high level and then estimating the remaining parameters. Here too, we are not able to estimate the model while simultaneously generating sensible behavior on the real side of the model as well as obtaining a high Sharpe-ratio.

The theoretical framework in this paper is taken from Lettau (1999). He presents closed-form solutions for risk premia of equity and long real bonds, the Sharpe-ratio as well for the process of the risk-free interest rates in the log-linear RBC model of Campbell (1994). These equations can be used as additional moment restrictions in the estimation of the RBC model. The advantage of the log-linear approach is that the

¹We use the 30-day rate to keep inflation uncertainty at a minimum.

closed-form solutions for the financial variables can be directly used in the estimation algorithm. Given a set of parameter values (or their estimates) no additional numerical procedure to solve the model is necessary.² This reduces the complexity of the estimation substantially. Note that estimating the parameter of relative risk aversion requires an iterative method when the Sharpe-ratio condition is present. Solving the model numerically and estimate it iteratively would be infeasible in this case.

The estimation technique in this paper follows the Maximum Likelihood (ML) method in Semmler and Gong (1996a). However, the algorithm has to be modified to allow for a simultaneous estimation of the risk aversion parameter and the Sharpe-ratio. As time series data on real variables we employ the data set provided by Christiano (1988). The estimation is conducted through a numerical procedure that allows us to iteratively compute the solution of the decision variables for given parameters and to revise the parameters through a numerical optimization procedure so as to maximize the Maximum Likelihood function. As optimization algorithm we employ the simulated annealing as used in Semmler and Gong (1997). We introduce a diagnostic procedure which is closely related to Watson (1993) and Diebold, Ohanian and Berkowitz (1995) to test whether the moments predicted by the model at the estimated parameters can match moments of the actual macroeconomic time series. We then use the variance-covariance matrix from the estimated parameters to infer the intervals of the moment statistics and to study whether the actual moments derived from the sample data fall within this interval.

The remainder of the paper is organized as follows. In Section II we introduce the log-linearization of the baseline RBC model and the closed-form solutions for the financial variables as computed in Lettau (1999). Section III describes the estimation procedure. Section IV presents the estimation results for the different models of our RBC model. In Section V we interpret our results contrasting the asset market implications of our estimates to the stylized facts of the asset market. We compare the second moments of the time series generated from the model to the moments of actual time series. The Appendix contains some derivations.

²In one version of the model we specify a given Sharpe-ratio. In this case one nonlinear equation has to be solved in order to obtain the implied parameter of relative risk aversion.

2 The RBC Model and its Asset Pricing Implications

2.1 The Baseline RBC Model

In this paper we follow Campbell's (1994) version of the standard RBC model. We use the notation Y_t for output, K_t for capital stock, A_t for technology, N_t for normalized labor input and C_t for consumption. The maximization problem of a representative agent is assumed to take the form

$$\text{Max} E_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+i}^{1-\gamma}}{1-\gamma} + \theta \log(1 - N_{t+i}) \right] \quad (1)$$

subject to

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t \quad (2)$$

with Y_t given by $(A_t N_t)^\alpha K_t^{1-\alpha}$. The first order conditions of this maximizing problem are

$$C_t^{-\gamma} = \beta E_t \{ C_{t+1}^{-\gamma} R_{t+1} \} \quad (3)$$

$$\frac{1}{\theta(1 - N_t)} = \alpha \frac{A_t^\alpha}{C_t} \left(\frac{K_t}{N_t} \right)^{(1-\alpha)} \quad (4)$$

where R_{t+1} is the gross rate of return on investment in capital which is equal to the marginal product of capital in production plus undepreciated capital:

$$R_{t+1} \equiv (1 - \alpha) \left(\frac{A_{t+1} N_{t+1}}{K_{t+1}} \right)^\alpha + 1 - \delta. \quad (5)$$

We allow firms to issue bonds as well as equity. Since markets are complete real allocations will not be affected by this choice (i.e. the Modigliani-Miller theorem holds).

We denote the leverage factor (the ratio of bonds outstanding and total firm value) as ζ .³

At the steady state, the technology, consumption, output and capital stock all grow at a common rate $G = A_{t+1}/A_t$. Hence, (3) becomes

$$G^\gamma = \beta R \quad (6)$$

where R is the steady state of R_{t+1} . Using lower case letters for the corresponding variables in logs, (6) can further be written as

³See Appendix II how leverage affects the equity premium.

$$\gamma g = \log(\beta) + r. \quad (7)$$

This defines the relation among g, r, β and γ . In the rest of the paper we use g, r , and γ as parameters to be determined, the implied value for the discount factor can then be deduced from (7).

2.2 The log-linear Approximate Solution

There are different ways to solve the above dynamic optimization problem. We follow the log-linear approximation method which has also been used by King et al. (1988a,b) and Campbell (1994) among others. To apply this method, one first needs to detrend the variables so as to transform them into stationary forms. For a variable X_t the detrended variable x_t is assumed to take the form $\log(X_t / \bar{X}_t)$, where \bar{X}_t is the value of X_t on its steady state path.

One, therefore, can think of x_t as the variable of zero-mean deviation from the steady state growth path of X_t . In Appendix I, we provide a description of how to derive \bar{X}_t , which depends on the initial condition \bar{X}_1 . The advantage to use this method of detrending is that one can drop the constant terms in the decision rules. Therefore, some structural parameters may not appear in the decision rule and hence one need not estimate them.⁴

Assume that the technology shock, a_t , follows an AR(1) process:

$$a_t = \phi a_{t-1} + \varepsilon_t \quad (8)$$

with ε_t the i.i.d innovation and standard deviation σ_ε .

Campbell (1994) shows that the solution, using the loglinear approximation method, can be written as

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t \quad (9)$$

$$n_t = \eta_{nk} k_t + \eta_{na} a_t \quad (10)$$

and the law of motion of capital is

⁴This is different from other approximation methods, such as Semmler and Gong (1996a,b) where the detrending method does not permit to drop the constant terms. Therefore, all the parameters have to be estimated.

$$k_t = \eta_{kk}k_{t-1} + \eta_{ka}a_{t-1} \quad (11)$$

where $\eta_{ck}, \eta_{ca}, \eta_{nk}, \eta_{na}, \eta_{kk}$, and η_{ka} are all complicated functions of the parameters $\alpha, \delta, r, g, \gamma, \phi$ and \bar{N} (the steady state of value N_t). Campbell and Koo (1997) study the accuracy of log-linear approximations such as this and find that the approximation error is small relative to numerical solution methods.⁵

2.3 Asset Prices

The baseline RBC model as presented above has strong implications for asset prices. Lettau (1999) presents closed-form solutions for a variety of financial variables. His results can be summarized as follows. First, consider the riskfree interest rate. The Euler equation (3) written in log-form together with the process of log-consumption (9) implies the following AR(1) process for the riskfree interest rate r_t^f :⁶

$$r_t^f = \gamma \frac{\eta_{ck}\eta_{ka}}{1 - \eta_{kk}L} \varepsilon_{t-1} \quad (12)$$

where L is the lag operator. Matching this process implied by the model to the data will be the first additional asset market restriction introduced later on. The second asset market restriction will be the Sharpe-ratio which summarizes the risk-return tradeoff implied by the model. See Hansen and Jagannathan (1992) and Lettau and Uhlig (1999) for detailed descriptions of the importance of the Sharpe-ratio in evaluating asset prices generated by various models.

$$SR_t = \max_{\text{all assets}} \frac{E_t [R_{t+1} - R_{t+1}^f]}{\sigma_t [R_{t+1}]} \quad (13)$$

Since the model is log-linear and has normal shocks, the Sharpe-ratio can be computed in closed form as (see Lettau and Uhlig (1999) for more details):

$$SR = \gamma \eta_{ca} \sigma_\varepsilon \quad (14)$$

Lastly we will compute the risk premia of equity (EP) and long-term real bonds (LTBP). Lettau (1999) computes these premia based on the loglinear solution of the

⁵This conclusion is probably not robust once highly nonlinear components are added to the model, e.g. habit formation, see Campbell et al. (1997)

⁶For ease of notation we ignore unimportant constants in the following equations.

RBC model (9)-(11). Appendix II presents a short derivation of the following equations:

$$LTBP = -\gamma^2 \beta \frac{\eta_{ck} \eta_{ka}}{1 - \beta \eta_{kk}} \eta_{ca}^2 \sigma_\varepsilon^2 \quad (15)$$

$$EP = \left(\frac{\eta_{dk} \eta_{nk} - \eta_{da} \eta_{kk}}{1 - \beta \eta_{kk}} - \gamma \beta \frac{\eta_{ck} \eta_{kk}}{1 - \beta \eta_{kk}} \right) \gamma \eta_{ca}^2 \sigma_\varepsilon^2. \quad (16)$$

2.4 Some Stylized Facts

The Table 1 summaries some key facts on asset markets and real economic activity for the US economy. A successful model should be consistent with these basic moments of real and financial variables. In addition to the well-known stylized facts on macroeconomic variables, we will consider the performance along the lines of the following facts from asset markets.

	Std.Dev.	Mean
GNP	1.72	
Consumption	1.27	
Investment	8.24	
Labor Input	1.59	
T-Bill	0.86	0.19
SP 500	7.53	2.17
Equity Premium	7.42	1.99
Long Bond Premium	0.21	4.80
Sharpe Ratio		0.27

Note: Standard Deviations (Std.Dev.) for the real variables are taken from Cooley and Prescott (1995). The series are H-P filtered from trend. Asset market data are from Lettau (1999). All data are at quarterly frequency. Units are per cent per quarters. The Sharpe-ratio is the mean of equity premium divided by its standard deviation.

The Table 1 shows that the equity premium is roughly 2% per quarter. The Sharpe-ratio, calculated as indicated below Table 1, measures the risk-return trade-off, which equals 0.27 in post-war data. The standard deviation of the real variables reveal the

usual hierarchy in volatility with investment being most volatile and consumption the smoothest variable.

3 The Estimation Method

3.1 The Maximum Likelihood Estimation

The Maximum Likelihood (ML) estimation, proposed by Chow (1993a, b) for estimating a stochastic dynamic optimization model, employs an econometric model such as

$$By_t + \Gamma x_t = \varepsilon_t \quad (17)$$

where B is a $m \times m$ matrix; Γ is a $m \times k$ matrix; y_t is a $m \times 1$ vector of dependent variables; x_t is a $k \times 1$ vector of explanatory variables and ε_t is a $m \times 1$ vector of disturbance terms. Note that both B and Γ are complicated functions of structural parameters, denoted by ψ , that we want to estimate.

Suppose that there are T observations. Then the above equation can be re-written as

$$BY' + \Gamma X' = E' \quad (18)$$

where Y, X , and E are respectively $t \times m$; $T \times k$ and $T \times m$ matrices. Assuming normal and serially uncorrelated ε_t with the covariance matrix Σ , the concentrated log-likelihood function can be derived (see Chow, 1983, p. 170-171) as

$$\log L(\psi) = const + T \log |B| - \frac{T}{2} \log |\Sigma| \quad (19)$$

with the ML estimation of Σ given by

$$\hat{\Sigma} = T^{-1}(BY' + \Gamma X')(YB' + X\Gamma'). \quad (20)$$

The ML estimator of ψ , denoted by $\hat{\psi}$, is the one that maximizes $\log L(\psi)$ in (19). The asymptotic standard errors of the estimated parameters can be inferred from the following variance-covariance matrix of $\hat{\psi}$ (see Hamilton (1994):

$$E(\hat{\psi} - \psi)(\hat{\psi} - \psi)' = \left[-\frac{\partial^2 L(\psi)}{\partial \psi \partial \psi'} \right]^{-1} \quad (21)$$

3.2 The Numerical Estimation Procedure

Using the Maximum Likelihood estimation method to estimate a stochastic growth model of RBC type can be rather complex, for both B and Γ are complicated functions of ψ , the parameter vector that we want to estimate. Usually, one is unable to derive the first-order conditions to maximize (19) with respect to the parameters ψ . Consequently, searching a parameter space becomes the only possible way to find the optimum. Furthermore, using first-order conditions to maximize (19) may only lead to a local optimum which is quite possible in general since the system to be estimated is often nonlinear in parameters.

Usually the search process includes the following recursive steps:

- start with an initial guess on ψ and use an appropriate method of stochastic dynamic programming to derive the decision rules;
- use the state equations and the derived control equations to calculate the value of the objective function as represented by (19);
- apply some optimization algorithm to change the initial guess on ψ and start again with step one.

One finds that conventional optimization algorithms⁷ such as Newton-Raphson or related methods, may not serve our purpose well due to the possibility of multiple local optima as in our case. We thus employ a global optimization algorithm, called simulated annealing. The idea of the simulated annealing has been initially proposed by Metropolis et al. (1953) and later developed by Vanderbilt and Louie (1984), Bohachavsky et al. (1986) and Corana et al. (1987) for continuous variable problems. The algorithm operates through an iterative random search for the variables of an objective function within an appropriate space. It moves uphill and downhill with a varying step size to escape local optima. Eventually the step is narrowed so that the random search is confined to an ever smaller region when the global optimum is approached.

The simulated annealing⁸ has been tested by Goffe et al. (1991). For this test Goffe et al. (1991) compute a test function with two optima provided by Judge et

⁷For conventional optimization algorithm, see Appendix B of Judge et al (1985) and Hamilton (1994, ch. 5).

⁸For further explanation of the simulated annealing and an application in econometric estimations, see Semmler and Gong (1997).

al. (1985, p. 956-7). By comparing it with conventional algorithms, they find that out of 100 times conventional algorithms are successful 52-60 times to reach the global optimum while the simulated annealing is 100 percent efficient. We thus believe that the algorithm is very suitable for our purpose.

4 The Estimation

4.1 Model Parameters

The RBC model presented in section 2 contains seven parameters, $\alpha, \delta, r, g, \gamma, \phi$, and \bar{N} . Recall that the discount factor is determined in (7) for given values of g, r and γ . Of course, we would like to estimate as many parameters as possible, However, some of the free parameters have to be prespecified. The calculation of the technology shocks requires values for α and g . The parameters ϕ and \bar{N} have to be fixed in order to compute the steady state of the model and have to be specified to detrend the data. In this paper we use the standard values of $\alpha = 0.667$, $g = 0.005$, $\bar{N} = 0.3$. The parameter ϕ is estimated from (8) by OLS regression. This leaves the risk aversion parameter γ , the average interest rate r and the depreciation rate δ to be estimated. This strategy is similar to Christiano and Eichenbaum (1992). However, they fix the discount factor (and hence the average interest rate) and the risk aversion parameter without estimating them. In contrast, the estimation of these parameters is central to our strategy, as we will see shortly.

4.2 The Data Set

Many empirical studies of the RBC model, including Christiano and Eichenbaum (1992), require to redefine the existing macroeconomic data so that they are accommodated to the definition of the variables as defined in the model. For example, the data of labor effort is modified to overcome the possible measurement error as suggested by Prescott (1986). Further, it is suggested that not only private investment but also government investment and durable consumption should be counted as adding to the capital stock K_t . The government capital stock, the stock of durable consumption goods (as well as the inventory and the value of land, see Cooley and Prescott, 1995) are included in K_t . Consequently, the service generated from durable consumption goods and government capital stock should also appear in the definition of Y_t .

Since such data are not readily available, one has to compute them based on some assumptions. In this paper we shall use the data set as constructed by Christiano (1988) and used in Christiano and Eichenbaum (1992).⁹ The data set covers the period from the third quarter of 1955 through the fourth quarter of 1983 (1955.1-1983.4). We shall remark that the Christiano (1988) data set can match the real side of the economy better than the commonly used NIPA data sets (see Semmler and Gong 1996b). For the time series of the riskfree interest rate we use the 30-day T-bill rate to minimize unmodeled inflation risk.

4.3 Estimation Strategy

In order to analyze the role of the each additional restriction for the parameter estimates, we introduce the restrictions step-by-step. First, we constrain the risk aversion parameter to unity and use only moment restrictions of the real variables (i.e. (9)-(11)) so we can compare our results to those in Christiano and Eichenbaum (1992). The remaining parameters to be estimated are δ and r . We call this model 1 (M1). According to (17) the matrices for the ML estimation are

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -\eta_{ck} & 1 & 0 \\ -\eta_{cn} & 0 & 1 \end{bmatrix}, \Gamma = \begin{bmatrix} -\eta_{kk} & -\eta_{ka} & 0 \\ 0 & 0 & -\eta_{ca} \\ 0 & 0 & -\eta_{na} \end{bmatrix}, \quad (22)$$

$$y_t = \begin{bmatrix} k_t \\ c_t \\ n_t \end{bmatrix}, x_t = \begin{bmatrix} k_{t-1} \\ a_{t-1} \\ a_t \end{bmatrix}. \quad (23)$$

After considering the estimation with moment restrictions only for real variables, we add restrictions from asset markets one by one. We start by including the following moment restriction of the riskfree interest rate in the estimation while still keeping risk aversion fixed at unity:

$$E [b_t - r_t^f] = 0, \quad (24)$$

where b_t denotes the return on the 30-day T-bill and the riskfree rate r_t^f in the RBC model is given in (12). We refer to this version as model 2 (M2). In this case the matrices B and Γ and the vectors x_t and y_t can be written as

⁹We would like to thank them for making available to us their data set.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\eta_{nk} & 1 & 0 & 0 \\ -\eta_{nk} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -\eta_{kk} & -\eta_{ka} & 0 & 0 \\ 0 & 0 & -\eta_{ca} & 0 \\ 0 & 0 & -\eta_{na} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (25)$$

$$y_t = \begin{bmatrix} k_t \\ c_t \\ n_t \\ b_t \end{bmatrix}, \quad x_t = \begin{bmatrix} k_{t-1} \\ a_{t-1} \\ a_t \\ r_t^f \end{bmatrix}. \quad (26)$$

Model 3 (M3) uses the same moment restrictions as model 2 but leaves the risk aversion parameter to be estimated rather than fixed to unity.

Finally we impose that the RBC model should generate a Sharpe-ratio of 0.27 as measured in the data (see Table 1). We take this restriction into account in two different ways. First, as a shortcut, we fix the risk aversion at 50, a value suggested in Lettau and Uhlig (1999) since it generates a Sharpe-ratio of 0.27 using actual consumption data. Given this value, we estimate the remaining parameter δ and r . This will be called model 4 (M4). In the next version, model 5 (M5), we are simultaneously estimating γ and impose a Sharpe-ratio of 0.27. Recall that the Sharpe-ratio is a function of risk aversion, the standard deviation of the technology shock and the elasticity of consumption with respect to the shock and the elasticity of consumption with respect to the shock η_{ca} . Of course, η_{ca} is itself a complicated function of γ . Hence, the Sharpe-ratio restriction becomes

$$\gamma = \frac{0.27}{\eta_{ca}(\gamma)\sigma_\varepsilon}. \quad (27)$$

This equation provides the solution of γ , given the other parameters δ and r . Since it is nonlinear in γ , we, therefore, have to use an iterative procedure to obtain the solution. For each given δ and r , obtained by the simulated annealing, we first set an initial γ , denoted by γ_0 . Then the new γ , denoted by γ_1 , is calculated from (27), which is equal to $0.27/[\eta_{ca}(\gamma_0)\sigma_\varepsilon]$. This procedure is continued until convergence.

We summarize the different cases in Table 2, where we start by using only restrictions on real variables and fix risk aversion to unity (M1). We add the riskfree rate restriction keeping risk aversion at one (M2), then estimate it (M3). Finally we add

the Sharpe-ratio restriction, fixing risk aversion at 50 (M4) and estimate it using an iterative procedure (M5). For each model we also compute the implied values of the log bond and equity premium using (15) and (16).

Table 2. Summary of Models

model #	Estimated Parameter	Fixed Parameters	Asset Restrictions
M1	r, δ	$\gamma = 1$	none
M2	r, δ	$\gamma = 1$	riskfree rate
M3	r, δ, γ		riskfree rate
M4	r, δ	$\gamma = 50$	riskfree rate, Sharpe-ratio
M5	r, δ, γ		riskfree rate, Sharpe-ratio

5 The Estimation Results

Table 3 summarizes the estimation results for the first three models. Standard errors are in parentheses. Entries without standard errors are preset and hence are not estimated. Consider first model 1 which only uses restrictions on real variables. The depreciation rate is estimated to be just below 2% which close to Christiano and Eichenbaum's (1992) results. The average interest rate is 0.77% per quarter or 3.08% on an annual basis. However, the estimate is fairly imprecise. The implied discount factor computed from (7) is 0.9972. These results confirm the estimates in Christiano and Eichenbaum (1992) and Semmler and Gong (1996b). Adding the riskfree rate restriction in model 2 does not significantly change the estimates. The discount factor is slightly higher while the average riskfree rate decreases. However the implied discount factor now exceeds unity, a problem also encountered in Christiano, Hansen and Singleton (1992). Christiano and Eichenbaum (1992) avoid this by fixing the discount factor below unity rather than estimating it. Model 3 is more general since the risk aversion parameter is estimated instead of fixed at unity. The ML procedure estimates risk aversion to be roughly 2 and significantly different from log-utility. Adding the riskfree rate restrictions increases the estimates of δ and r somewhat. Overall, the model is able to produce sensible parameter estimates when moment restrictions for the riskfree rate are introduced.

Table 3. Summary of Estimation Results The standard errors are in parenthesis.

model #	δ	r	γ
M1	0.0189 (0.0144)	0.0077 (0.0160)	prefixed to 1
M2	0.0220 (0.0132)	0.0041 (0.0144)	prefixed to 1
M3	0.0344 (0.0156)	0.0088 (0.0185)	2.0633 (0.4719)

⁰The standard errors are in parenthesis.

While the implications of the RBC concerning the real macroeconomic variables are considered as fairly successful, the implications for asset prices are dismal. Table 3 computes the Sharpe-ratio as well as risk premia for a long term real bond and equity using (14)-(16). Note that these variables are not used in the estimation of the model parameters. The leverage factor ζ is set to $2/3$ for the computation of the equity premium.¹⁰

Table 4. Asset Pricing Implications

model #	SR	LT BPrem	EqPrem
M1	0.0065	0.000%	-0.082%
M2	0.0065	-0.042%	-0.085%
M3	0.0180	-0.053%	-0.091%

Table 4 shows that the RBC is not able to produce sensible asset market prices when the model parameters are estimated from restrictions derived only from the real side of the model (or, as in M3, adding the risk-free rate). The Sharpe-ratio is too small by a factor of 50 and both risk premia are too small as well, even negative for certain cases. Introducing the riskfree rate restriction improves the performance only a little bit. Next, we will try to estimate the model by adding the Sharpe-ratio moment restrictions.

Model 4 fixes risk aversion at 50. As explained in Lettau and Uhlig (1999), such a high level of risk aversion has the potential to generate reasonable Sharpe-ratios in consumption CAPM models. The question now is how the moment restrictions of the real variables are affected by such a high level of risk aversion. The first row of Table 5 shows that the resulting estimates are not sensible. We constrain the estimates to lie between 0 and 1 in the estimation procedure. Table 5 shows that estimates for the depreciation factors and the steady-state interest rate converge to the prespecified

¹⁰This value is advocated in Benninga and Protopapadakis (1990).

constraints. The estimation does not settle down to an interior optimum. In other words, the real side of the RBC does not yield reasonable results when risk aversion is 50. High risk aversion implies a low elasticity of intertemporal substitution so that agents are very reluctant to change their consumption over time. The RBC model is not capable of matching the volatility of consumption in the data when risk aversion is exogenously fixed at a high level.

Trying to estimate risk aversion while matching the Sharpe-ratio gives similar results. It is not possible to estimate the RBC model with simultaneously satisfying the moment restrictions from the real side of the model and the financial side, as shown in the last row in Table 5. Again the parameter estimates do converge to prespecified constraints. The depreciation rate converges again to unity as does the steady-state interest rate r . The point estimate of the coefficient of relative risk aversion is high (60). The reason is of course that a high Sharpe-ratio requires high risk aversion. The tension between the Sharpe-ratio restriction and the real side of the model causes the estimation to fail. It demonstrates again that asset pricing relationships are fundamentally incompatible with the RBC model.

Table 5. Matching the Sharpe-Ratio

model #	δ	r	γ
M4	1	0	prefixed to 50
M5	1	1	60

6 The Evaluation of Predicted and Sample Moments

Next we provide a diagnostic procedure to compare the second moments predicted by the model with the moments implied by the sample data. Our objective here is to ask whether our RBC model can predict the actual moments of the time series for both the real and asset market. The moments are revealed by the spectra at various frequencies. We remark that a similar diagnostic procedure can also be found in Watson (1993) and Diebold et. al (1995).

Given the observations on k_t, a_t, k_{t-1} and a_{t-1} (and fixed and estimated parameters of our log-linear model), the predicted c_t and n_t and k_t can be constructed from the right hand side of (9)-(11) with k_t, a_t, k_{t-1} and a_{t-1} to be their actual observations. We now consider the possible deviations of our predicted series from the sample series.

We hereby employ our most reasonable estimated model 3. We can use the variance-covariance matrix of our estimated parameters to infer the intervals of our forecasted series hence also the intervals of the moment statistics that we are interested in.

Figure 1 presents the Hodrick-Prescott (HP) filtered actual and predicted time series on consumption, labor effort, riskfree rate and equity return. As other studies have shown (see Semmler and Gong 1996a for instance) the consumption series can somewhat be matched where as the variation in the labor series as well as in the risk-free rate and equity return cannot be matched. The insufficient match of the latter three series are confirmed by Figure 2 where we compare the spectra calculated from the data samples to the intervals of the spectra predicted, at 5% significance level, by the models.

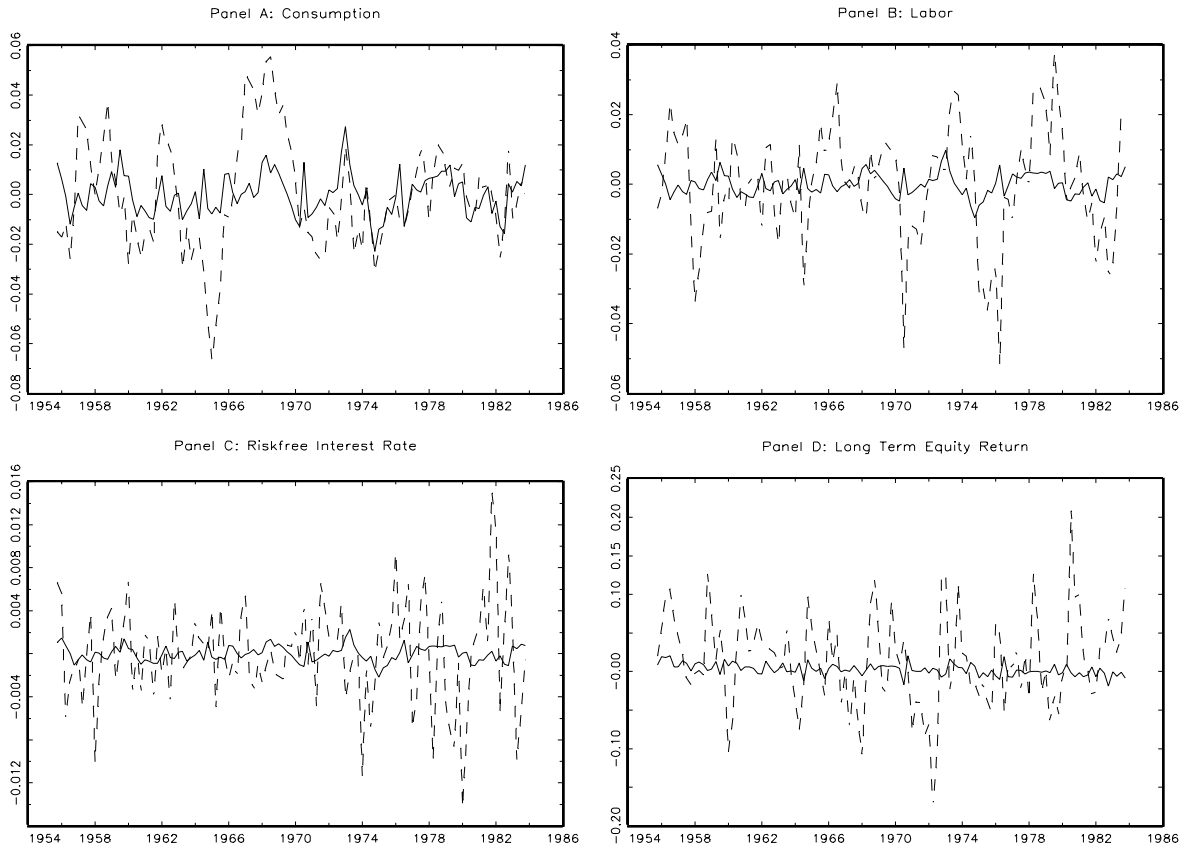


Figure 1: Predicted and Actual Series: Solid Lines for Actual Series, Dotted Lines for Predicted Series

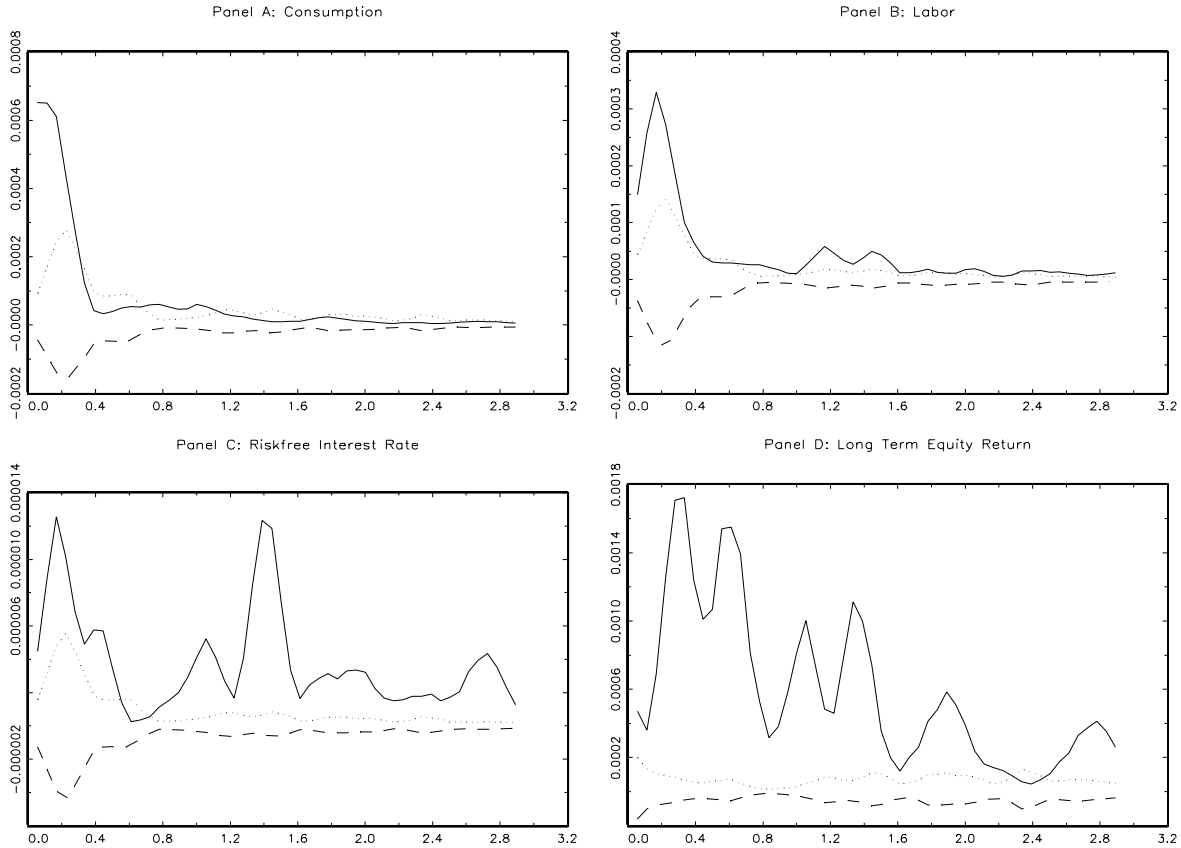


Figure 2: The Second Moment Comparison: Solid Line for Actual Moments, Dashed and Dotted Lines for the Intervals of Predicted Moments

A good match of the actual and predicted second moments of the time series would be represented by the fact that the solid line falls within the interval of the dashed and dotted lines. In particular the time series for labor effort, riskfree interest rate and equity return fail to do so.

7 Conclusions

Asset prices contain valuable information about intertemporal decision making of economic agents. This paper estimates the parameters of a standard RBC model taking the asset pricing implications into account. We introduce model restrictions based on

asset pricing implications in addition to the standard restrictions of the real variables and estimate the model by using ML method. We use the risk free-interest rate and the Sharpe-ratio in matching actual and predicted asset market time series data and compute the implicit risk premia for long real bonds and equity. To pursue the estimation we introduce asset pricing restrictions step-by-step to clearly demonstrate the effect of each new restriction. First, we estimate the model using the real variables. For this case we obtain estimates close to those in Christiano and Eichenbaum (1992) and Semmler and Gong (1996b). Second, we include the risk-free interest rate into the model to be matched to actual data. The 30-day T-bill rate is used in the estimation procedure. The estimated parameters are fairly close to the estimates of the model based only on real variables. Third, we add the Sharpe-ratio (as a general measure of the risk-return trade-off) as a second asset market restriction. It does not depend on any specific dividend process of some asset. Though the inclusion of the risk-free interest rate as a moment restriction can produce sensible estimates, the computed Sharpe-ratio and the risk premia of long real bonds and equity are in general counterfactual. The computed Sharpe-ratio is too low while both risk premia are small and even negative. More over, the attempt to match the Sharpe-ratio in the estimation process can hardly generate sensible estimates. Finally, the second moments of labor effort, risk-free interest rate and long term equity return predicted by the model do not match well the corresponding moments of the sample economy.

We conclude that the baseline RBC model cannot match the asset market restrictions, at least with the standard technology shock, constant relative risk aversion (CRRA) utility function and no adjustment costs. We thus may have to look at some extensions of the model such as technology shocks with a greater variance, other utility functions, for example, utility functions with habit formation, and adjustment costs. The latter line of research has been proposed by Boldrin, Christiano and Fisher (1996).

Appendix I: The Detrending Procedure

This appendix describes how we detrend the data series X_t into x_t by applying $x_t = \log(X_t/\bar{X}_t)$ where $\bar{X}_t = (1+g)^{t-1}\bar{X}_1$. Apparently, for the given g , the series x_t depends on \bar{X}_1 , which can be regarded as the initial \bar{X}_t . Our objective is to find the \bar{X}_t such that the mean of x_t is equal to 0. In other words,

$$\begin{aligned} (1/T) \sum_{i=1}^T \log(X_t/\bar{X}_t) &= (1/T) \sum_{i=1}^T \log(X_t) - (1/T) \sum_{i=1}^T \log(\bar{X}_t) \\ &= (1/T) \sum_{i=1}^T \log(X_t) - (1/T) \sum_{i=1}^T \log(\bar{X}_1) - (1/T) \sum_{i=1}^T \log[(1+g)^{t-1}] \\ &= 0 \end{aligned}$$

Solving the above equation for \bar{X}_1 , we obtain

$$\bar{X}_1 = (1/T) \exp \left\{ \sum_{i=1}^T \log(X_t) - \sum_{i=1}^T \log[(1+g)^{t-1}] \right\}$$

Appendix II: Asset Prices in the Log-Linear Model

1. Risk-Free-Rate

From the Lucas asset pricing formula we have (ignoring constants): ¹¹.

$$r_{t+1}^f = \gamma E_t \Delta c_{t+1}$$

To obtain the time-series for c_{t+1} , note that the RBC model yields

$$k_t = \eta_{kk} k_{t-1} + \eta_{ka} a_t$$

and

$$a_t = \phi a_{t-1} + \epsilon_t = \frac{\epsilon_t}{1 - \phi L}$$

with L the lag-operator. Hence, we get

$$k_t = \frac{\eta_{ka}}{(1 - \eta_{kk} L)(1 - \phi L)} \epsilon_t,$$

which is an AR(2). For c_t we find

$$\begin{aligned} c_t &= \eta_{ck} k_{t-1} + \eta_{ca} a_t \\ &= \frac{\eta_{ck} \eta_{ka}}{(1 - \eta_{kk} L)(1 - \phi L)} \epsilon_{t-1} + \frac{\eta_{ca}}{(1 - \phi L)} \epsilon_t \\ &= \frac{\eta_{ca}}{(1 - \phi L)(1 - \eta_{kk} L)} \epsilon_t + \frac{\eta_{ck} \eta_{ka} - \eta_{ca} \eta_{kk}}{(1 - \eta_{kk} L)(1 - \phi L)} \epsilon_{t-1} \\ &= \frac{\eta_{ca} + (\eta_{ck} \eta_{ka} - \eta_{ca} \eta_{kk}) L}{(1 - \phi L)(1 - \eta_{kk} L)} \epsilon_t \end{aligned}$$

¹¹The conditional variance of consumption growth in this model is constant and therefore ignored in the following equations

In other words c_t follows an ARMA (2,1) process. For simplicity we assume that a_t follows a random walk, i.e. $\phi = 1$. Hence

$$\Delta c_t = \frac{\eta_{ca} + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})L}{1 - \eta_{kk}L} \epsilon_t$$

It is easy to show that

$$\begin{aligned} E_t \Delta c_{t+1} &= \eta_{ca} \frac{\eta_{kk} + \frac{\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{ka}}{\eta_{ca}}}{1 - \eta_{kk}L} \epsilon_t \\ &= \frac{\eta_{ck}\eta_{ka}}{1 - \eta_{kk}L} \epsilon_t \end{aligned}$$

Hence, the risk-free rate follows an AR(1) process:

$$r_{t+1}^f = \gamma \frac{\eta_{ck} \cdot \eta_{ka}}{1 - \eta_{kk}L} \epsilon_t \quad (A.1) \quad (28)$$

2. Long-term assets (if Δc_t is an ARMA(1,1) process)

Campbell and Shiller (1988) have shown that the unexpected return can be written as

$$\begin{aligned} r_{t+1} - E_t r_{t+1} &= (E_{t+1} - E_t) \sum_{j=1}^{\infty} \beta^{j-1} \Delta d_{t+1} \\ &\quad - (E_{t+1} - E_t) \sum_{j=2}^{\infty} \beta^{j-1} r_{t+j} \end{aligned}$$

where d_t is the dividend in period t .

For a long real bond: $d_t = 1 \quad \forall t$. Hence using (A.1) and the fact that risk premia are constant, we obtain

$$\begin{aligned} r_{t+1} - E_t r_{t+1} &= -E_{t+1} \left[\beta r_{t+2} + \beta^2 r_{t+3} + \dots \right] \\ &\quad + E_t \left[\beta r_{t+2} + \beta^2 r_{t+3} + \dots \right] \\ &= -\gamma \beta \frac{\eta_{ck}\eta_{ka}}{1 - \beta\eta_{kk}} \epsilon_{t+1} \end{aligned}$$

Equity dividends equal the rental rate of capital. For a Cobb-Douglas production function log-dividends are proportional to the difference between log-output and log-capital. Let $\lambda = \rho \frac{\bar{Y}}{\bar{K} \bar{D}}$ and $\bar{D} = (1 - \zeta)(\rho \frac{\bar{Y}}{\bar{K}} + 1 - \delta)$. Then

$$\begin{aligned} d_t &= \lambda(y_t - k_{t-1}) \\ &= \lambda(\eta_{yk} k_{t-1} + \eta_{ya} a_t - k_{t-1}) \\ &= \lambda(\eta_{yk} - 1)k_{t-1} + \eta_{ya} a_t \\ &= \frac{\eta_{da} + (\eta_{dk}\eta_{ka} - \eta_{da}\eta_{kk})L}{(1 - \phi L)(1 - \eta_{kk}L)} \epsilon_t \end{aligned}$$

where $\eta_{dk} = \lambda(\eta_{yk} - 1)$ and $\eta_{da} = \lambda\eta_{ya}$.

Straightforward calculations show that

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \beta^j \Delta d_{t+j} = \frac{\eta_{dk}\eta_{ka} - \eta_{da}\eta_{kk}}{1 - \beta\eta_{kk}} \epsilon_{t+1}$$

Hence, for long real equity we have

$$r_{t+1} - E_t r_{t+1} = \left(\frac{\eta_{dk}\eta_{ka} - \eta_{da}\eta_{kk}}{1 - \beta\eta_{kk}} - \gamma\beta \frac{\eta_{ck}\eta_{ka}}{1 - \beta\eta_{kk}} \right) \epsilon_{t+1}$$

As shown in Lettau (1999) this results in the following risk premia:

$$LTBP = -\gamma^2 \beta \frac{\eta_{ck}\eta_{ka}}{1 - \beta\eta_{kk}} \eta_{ca}^2 \sigma_\epsilon^2$$

$$EP = \left(\frac{\eta_{dk}\eta_{kk} - \eta_{da}\eta_{kk}}{1 - \beta\eta_{kk}} - \gamma\beta \frac{\eta_{ck}\eta_{kk}}{1 - \beta\eta_{kk}} \right) \gamma \eta_{ca}^2 \sigma_\epsilon^2$$

where LTBP and EP are the long term real bond premium and equity premium respectively.

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