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Solving Ecological Management Problems Using Dynamic Programming

by

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Abstract

We study an ecological management problem where economic agents' activities interact with the dynamics of natural resources. We follow the work by Brock and his various co-authors and use the example of a shallow lake which is subject to pollution due to phosphorous loading. Low loading preserves resilience of the ecosystem (oligotrophic state) where high loading may lead to the deterioration of the ecosystem (eutrophic state). Welfare is calculated from expected discounted net benefits: benefits accrue to agricultural interests from activities that result in phosphorous loading and costs – resulting from deterioration of water quality – accrue the enjoyers of the lake. The interaction of the dynamic decision problem maximizing welfare and the dynamics of the ecosystem admits multiple equilibria, thresholds and complicated global dynamics. We consider instruments of a regulatory agency, for example, state dependent tax rates that may help to maintain and to enhance resilience by enlarging the domain of attraction of the low pollution equilibrium (oligotrophic state) or make it the sole attractor. The complicated global dynamics is analytically studied by using the Hamilton-Jacobi-Bellman (HJB) equation and numerically solved through dynamic programming.

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1 Introduction

Research in the last decade has shown that the relationship between the environment and economic activity is a rather complex one. Recent studies on ecological management problems reveal that the interaction of human activity and the ecological dynamics generates intricate dynamics which we need to understand in order to undertake welfare analysis and to make policy decisions.

To demonstrate the existence of such a complexity, researchers often have focused on a lake management problem. The lake model can be viewed as a symbolic model. Here the problem of the behavior of a shallow lake is studied that is subject to pollution by phosphorous. The model employed here reflects the characteristics of many ecological systems. They imply the integration of two aspects; the dynamic decision problem maximizing welfare and the dynamics of the ecosystem. This often generates properties of complex dynamics such as multiple equilibria, history dependence, thresholds between multiple attractors and a discontinuous policy function.

Carpenter, Ludwig and Brock (1999) and Brock and Starrett (1999) propose a deterministic version of an optimal management problem for ecosystems where there is a phosphorous loading into the lake due to economic activities . This affects the stock of phosphorous in the lake. When the stock of phosphorous becomes too high, internal positive feedback mechanisms start to impair the ecosystem's ability to absorb and biodegrade the loading. Welfare is defined as sum of utilities from loading rate and disamenities from degraded lake quality proportional to the level of the stock variable. The management can measure the stock and can control the loading as function of the stock. The management chooses these loadings to maximize welfare of the conflicting interests of polluters and lake users. The model has one state variable and its dynamics is simplified as a one dimensional ordinary differential equation.

The work by Carpenter, Ludwig and Brock (1999) and Carpenter, Brock and Hanson (1999) is extended in Ludwig, Carpenter and Brock (2002) to give a more precise account of the internal loading due to changing levels of dissolved phosphorous in the lake sediment in a stochastic environment. Using two state variables, the phosphorous in the water and the phosphorous in the lake sediment, the optimal loading is derived as a function of those two state variables. They argue that simple policies that neglect dynamics of the phosphorous in the sediments are inadequate unless the time horizon is short and the dynamics are slow. They also mention that a stochastic model is essential if there are substantial random fluctuations in loading. It is important particularly when the optimal solution to the system shows history dependence where the lake possibly flips between two attractors due to the disturbance. Thus they argue that the management problem should be described as a stochastic two dimensional state variable problem.

Another stream of literature to analyze this problem is using a game theoretic approach with N communities as in Mäler, Xepapadeas and de Zeeuw (1999) and Dechert and Brock (2000). They focus on the Nash equilibria solutions to the dynamic lake game. Dechert and Brock (2000) uses discrete dynamic programming to obtain a numerical solution. They show that the social optimum loading is always less than the total loading in the dynamic game. As N becomes large, the divergence between the dynamic game solution and the social optimum becomes extreme and also complex dynamics can appear in between.

Mäler et al. (1999) provide the optimal management solution in the same context but investigate whether it is possible to induce an optimal management in case of common use of the lake by means of a tax rate. Their results shows that, for a small number of communities, a constant tax on phosphorous loading can induce optimal behavior in the long run, but for a large number it depends again on the history of the lake.

Perrings (1999) suggests, in the shallow lake example, that an aim of decision maker should be to maintain the system in a desirable state. The aim of policy should be to assure the sustainability of desirable states by assuring the resilience of the system in those states since there are significant costs to being trapped in a locally stable eutrophic state. As Carpenter, Brock and Hanson (1999) show, eutrophication causes the loss of the potential benefits of fresh water, including consumption of water by the surrounding population, irrigation, industrial uses, and recreation. Once a lake reaches an eutrophic state, it is very costly to reduce the phosphorous level and restore clean water. However, note that disamenities from degraded lake quality is already taken into account in our optimization problem. It means that the optimal management does not necessarily choose the path to the oligotrophic state. It happens when the society does not care much about lake quality and when the path going to the eutrophic state generates a higher value than the path going to the oligotrophic state for a given initial phosphorous state.

However, we still could argue that the eutrophic state is not desirable and there is enough reason for a neutral institution, for example, a regulatory agency to actively intervene to restore a resilient oligotrophic state. There are several reasons that one should intervene into the system dynamics. First, we neglect further negative externalities from the lake locked into eutrophic state, e.g. the unexpected loss of diversity of species in a food chain. The collapse of a food chain which starts from the eutrophic state of the lake may cause an irreparable crisis. However, if we incorporate the dynamics of each species and their interaction into our model, its theoretical treatment will be hopeless due to the complexity of the interaction effects. Those unexpected negative externalities which we neglect in the model can be a reason for an active intervention.

Another reason is that the optimization is done only from the view point of the current generation. However, it is impossible to know the future generation's preference. We also know, as aforementioned, that once a lake reaches an eutrophic state, it can be very costly to restore clean water and future generations will have to pay a high cost for restoring such a lake when they prefer an oligotrophic state. It seems that, for the decision maker, it is fundamental to maintain the desirable function of the ecosystem where the ecosystem keeps the capability to absorb and biodegrade loading. The ecosystem should also be a resilient one about the oligotrophic state.

In this paper, we rather go back to the simplest deterministic version by Carpenter, Ludwig and Brock (1999) and Brock and Starrett (1999). They detect the existence of multiple equilibria. When multiple equilibria arise, our interest will be to study the global dynamics and its policy implication. It has been recognized by Brock and Starrett (1999) that one could obtain three different scenarios on global dynamics in the simplest variant of the model with three equilibria; (1) LSS (Lower Steady State): the stable manifold associated with the low pollution equilibrium is dominant for any given initial condition, (2) Skiba: there is a threshold (Skiba Point) which separates different domains of attraction, and (3) USS (Upper Steady State): the stable manifold associated with the high pollution equilibrium is dominant for any given initial condition. We study those three scenarios analytically by using the Hamilton-Jacobi-Bellman (HJB) equation and numerically by dynamic programming, which permits us to detect a threshold, if it exists, and to derive a policy function which may be discontinuous.

For the purpose of an ecological policy, we are in particular interested in the scenario (1), the situation for which for any starting point the optimal policy will converge on the stable manifold to the low pollution equilibrium and make the ogliotrophic state a global atractor. We will explore the possibility of enlarging its domain of attraction by introducing a tax rate on the phosphorous loading whereby the tax rate will depend on the stock of phosphorous. Such a tax rate will make the lake dynamics converge to the low pollution equilibrium.

Overall, our interest is whether we can create a resilient system with oligotrophic optimal state by a tax rate on loading imposed by the manager of the lake or some regulatory agency. Our definition of desirable resilient system is where we obtain the scenario LSS – the stable manifold associated with the low pollution equilibrium is dominant for any feasible initial state. Our tax rate that is increasing in the stock of phosphorous is successful in the sense that we indeed can convert the scenarios of both Skiba and USS into a scenario of LSS. If the decision maker is successful to keep the latter scenario, for any initial state of phosphorous, the path to the oligotrophic state will generate the maximum value and it will be actually chosen by the decision maker. Since the history dependent property disappears and the size of the stability domain corresponding to oligotrophic attractor has been enlarged, the system may be able to absorb and biodegrade loading to a certain degree even if external shocks occur to the system.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 studies the problem by using the HJB equation and shows the analytical procedure. Section 4 introduces a tax rate into the model and applies again the HJB equation to this new problem. Section 5 presents results using dynamic programming to compute the global dynamics of the model without and with tax rate. Section 6 concludes the paper.

2 The Model

Following Brock and Starrett (1999), we consider a community of individuals who share the lake and its watershed. These individuals have conflicting interests. There are the interests of the affectors who obtain some benefits arising from phosphorous loading. While the affectors damage the ecosystem, the lake, the enjoyers face disamenities from degraded quality of the lake. The decision maker can be interpreted as a regulatory institution that coordinates those interests.

The objective of the decision maker is:

$$
\begin{aligned}
W(x) &= Max \int_0^{\infty} e^{-rt} U(x,a) dt \qquad (1)\\
&= Max \int_0^{\infty} e^{-rt} (u(a) - kc(x)) dt
\end{aligned}
$$

subject to

$$
\begin{aligned}
\dot{x} = a - \delta x - p(x),\ x(0) = x_0 \\
u' > 0,\ u'(0) = +\infty,\ u'' \leq 0 \\
c(0) = 0,\ c' \geq 0,\ c'(0) = 0,\ c'' \geq 0 \\
p(0) = 0,\ p' \geq 0,\ p'(0) = p'(+\infty) = 0 \\
x_m = arg\ max\ p',\ p'' > 0 \text{ for } x < x_m,\ p'' < 0 \text{ for } x > x_m
\end{aligned}
$$

More precisely, we define $U(x,a) = U_1(a) + U_2(x)$, $U_1(a) = u(a)$, the utility of the affectors, $U_2(x) = -kc(x)$, the utility of the enjoyers, and k, the relative importance of affectors and enjoyers.

The dynamic behavior of the lake is represented by one dimensional ordinary differential equation:

$$
\begin{aligned}
\dot{x} = a - \delta x + p(x) \equiv G(x,a) \qquad (2)\\
x(0) = x_0
\end{aligned}
$$

where x is the stock of phosphorous, a, the phosphorous input, δ, the rate of loss per unit of the stock, $p(x)$, the internal loading (convex-concave function).

We restrict our analysis by assuming

$$
p(x) < \delta x \text{ for all } x \geq 0 \qquad (3)
$$

so that functional reversibility of the lake state is assured with strictly positive a for all positive x.

For more specific results, we use the following specifications

$$u(a) = \alpha a^{\sigma};\ 0 < \sigma < 1 \tag{4}$$

$$c(x) = \beta x^2 \tag{5}$$

$$p(x) = \frac{mx^{\rho}}{n^{\rho} + x^{\rho}}. \tag{6}$$

Choosing the parameters $a = 2,\ \sigma = \frac{1}{2},\ \beta = \frac{1}{2},\ m = n = 1,\ \rho = 2$ for convenience we can rewrite the functions (4), (5) and (6) as:

$$u(a) = 2a^{\frac{1}{2}} \tag{7}$$

$$c(x) = \frac{1}{2}x^2 \tag{8}$$

$$p(x) = \frac{x^2}{1 + x^2}. \tag{9}$$

3 The HJB-Equation

We use the Hamilton-Jacobi-Bellmann (HJB) equation to study the analytical solution of this problem.[1] The HJB-equation for the present model has the form:

$$\begin{aligned} rV(x) &= \max_a [U(x,a) + V'(x)G(x,a)] \\ &= \max_a [u(a) - kc(x) + V'(x)(a - \delta x + p(x))] \end{aligned} \tag{10}$$

Depending on the parameters, there may exist multiple equilibria with different types of global dynamics. However, we can analyze the global dynamics by computing the value function and detect the dominant equilibrium or thresholds if they exist. We undertake three steps to compute the value function.

Step 1: Compute the steady states for the stationary HJB-equation.

[1] The subsequent methodology is worked out in more detail in Semmler and Sieveking (2000).

If e is an equilibrium then

$$G(e,a) \equiv a - \delta e + p(e) = 0 \tag{11}$$

and

$$rV(e) = u(a) - kc(e) \tag{12}$$

$$V'(e) = \frac{kc'(e)}{r + \delta - p'(e)}. \tag{13}$$

Because the equilibrium e satisfies

$$rV(e) = \max_a [u(a) - kc(e) + V'(e)(a - \delta e + p(e))], \tag{14}$$

substituting (11) and (12) into (13) yields

$$u(a) - kc(e) = \max_a \left[u(a) - kc(e) - \frac{kc'(e)}{r + \delta - p'(e)}(a - \delta e - p(e)) \right]. \tag{15}$$

Solving $\frac{d}{da}[\cdot] = 0$ gives

$$u'(a) - \frac{kc'(e)}{r + \delta - p'(e)} = 0. \tag{16}$$

From the specific functions of (7) and (8) we obtain

$$a^{\frac{1}{2}} - \frac{ke}{r + \delta - \frac{2e}{(1+e^2)^2}} = 0 \tag{17}$$

or

$$a = \frac{\left(r + \delta - \frac{2e}{(1+e^2)^2} \right)^2}{k^2 e^2} \tag{18}$$

Inserting this condition into the steady state condition (11), we have

$$\frac{\left(r + \delta - \frac{2e}{(1+e)^2} \right)^2}{k^2 e^2} - \delta e + \frac{e^2}{1 + e^2} = 0 \tag{19}$$

or

$$\{\delta k^2 e^3 - (r+\delta)^2\}(1+e^2)^4 - k^2 e^4 (1+e^2)^3 + 4(r+\delta)e(1+e^2)^2 - 4e^2 = 0 \quad (20)$$

Solving (20) for e, we can easily obtain multiple steady states. Note that we are only interested in the non-negative steady states, $e \geq 0$, for economic reason. The second-order necessary condition is satisfied by

$$\frac{d^2}{da^2}[\cdot] = u''(a) = -\frac{1}{2}a^{-\frac{3}{2}} < 0. \quad (21)$$

Step 2: Solve the dynamic HJB-equation starting from the equilibrium candidates.

Using the satisfactory HJB-equation again we obtain

$$\begin{aligned} rV(e) &= \max_a [u(a) - kc(e) + V'(e)(a - \delta e + p(e))] \quad (22) \\ &= \max_a \left[2a^{\frac{1}{2}} - \frac{1}{2}ke^2 + V'(e)\left(a - \delta e + \frac{e^2}{1+e^2}\right)\right]. \end{aligned}$$

Solving $\frac{d}{da}[\cdot] = 0$ gives $a^{-\frac{1}{2}} + V'(e) = 0$ or

$$a^{\frac{1}{2}} = -\frac{1}{V'(e)} \text{ and } a = \frac{1}{V'(e)^2}. \quad (23)$$

Substituting (23) into (26) gives

$$rV(e) = -2\frac{1}{V'(e)} - \frac{1}{2}ke^2 + V'(e)\left(\frac{1}{V'(e)^2} - \delta e + \frac{e^2}{1+e^2}\right). \quad (24)$$

therefore

$$V'(e)^2 + \frac{rV(e) + kc(e)}{\delta e - p(e)}V'(e) + \frac{1}{\delta e - p(e)} = 0. \quad (25)$$

Then, by solving (25) for $V'(e)$, we obtain an ordinary differential equation in V with candidates of steady states as initial conditions:

$$V'(e) = -\frac{rV(e) + kc(e)}{2(\delta e - p(e))} \pm \sqrt{\left(\frac{rV(e) + kc(e)}{2(\delta e - p(e))}\right)^2 - \frac{1}{\delta e - p(e)}} \quad (26)$$

Using this information for the solution of V we get

$$V'(x) = -\frac{rV(x) + kc(x)}{2(\delta x - p(x))} - \sqrt{\left(\frac{rV(x) + kc(x)}{2(\delta x - p(x))}\right)^2 - \frac{1}{\delta x - p(x)}} \text{ for } x \geq e \tag{27}$$

and

$$V'(x) = -\frac{rV(x) + kc(x)}{2(\delta x - p(x))} + \sqrt{\left(\frac{rV(x) + kc(x)}{2(\delta x - p(x))}\right)^2 - \frac{1}{\delta x - p(x)}} \text{ for } x < e \tag{28}$$

with candidates of steady states obtained from (20) as initial conditions. Local value functions are obtained about those steady states. Note that, from the assumption (3), $p(x) < \delta x$, $V'(x) < 0$ for any x since the stock of phosphorous, x, affects the value function in our model.

Step 3: Solve the global value function.

We can compute the global value function for the original problem by

$$V(x) = max V_i(x). \tag{29}$$

Global value function analysis is used to derive a unique policy function. When multiple equilibria rise, we generally have multiple paths to the different equilibria. Each of those paths generates a different currently discounted value and this information is summarized in the global value function. Obtaining the global value function, we undertake a path selection to obtain a unique policy function. As we will demonstrate below depending on the parameter set, the following three different scenarios of global dynamics can typically arise. (1) LSS (Lower Steady State): the stable manifold associated with the low pollution equilibrium is dominant for any given initial condition, (2) Skiba: there is a threshold (Skiba point) which separates different domains of attraction, and (3) USS (Upper Steady State): the stable manifold associated with the high pollution equilibrium is dominant for any given initial condition. When the Skiba case appears, the information on the study of the global value function is crucial in order to detect the exact point of the threshold, the so-called Skiba point. We later, in Section 5, present this analysis in more detail using dynamic programming with numerical values. Our simulations will focus on the changes of the scenarios between LSS, Skiba and USS due to the variation of the parameter k, the relative importance of affectors and enjoyers.

4 The Model with Taxation

Our results from the above control model without taxation suggests that we will generally face, when multiple equilibria arise, one of the above three scenarios of the global dynamics: LSS, Skiba or USS, depending on the parameter set. It implies that the optimal management does not necessarily choose the path to the oligotrophic steady state. When we are in the scenario of Skiba there exists a threshold (Skiba point) which separates different domains of attraction, the domain of the oligotrophic steady state (low pollution state) and the domain of the eutrophic steady state (high pollution state). For any given initial point of phosphorous stock above the threshold the optimal control will choose the path to the eutrophic steady state while another path to the oligotrophic steady state will be chosen only when the given initial point of phosphorous stock is below the threshold. In other words, the Skiba case has the property of history dependence. On the other hand, when we are in the scenario of USS, the optimal policy will be to choose the path to the eutrophic steady state irrespective of the initial phosphorous stock.

When we undertake simulations, in Section 5, by varying the parameter k, the relative importance of affectors and enjoyers, it turns out that for lower k, the optimal control is likely to choose the path to the eutrophic steady state.Note that k is given in our model and the optimal path is consistent with the preferences of both affectors and enjoyers. However, a point to notice is that k only presents their preference for the condition of the lake over the payoff from the economic activity. Yet, in reality, there exist significant costs to a lake being trapped in a eutrophic state. Giving only attention to disamenities from an eutrophic state of the lake seems not enough. For example, we may have, as aforementioned, negative externalities from the lake locked into an eutrophic state such as loss of diversity of species through a food chain which is caused by a pernicious condition of the lake. The condition of the lake affects the dynamics of the plants and animals living around the lake and the changes of the species dynamics will cause a further change of the whole ecosystem. Yet, the effects from those negative externalities and interactions are not taken into account in our model. We thus have to admit that there is a limit in incorporating all the complexity of the interaction effects in our theoretical treatment.

Another problem, as above mentioned, is the one stemming from the different preferences between current and future generations and the problem of intertemporal justice.[2] We will later, in the numerical examples, show the fact that once a lake reaches a eutrophic state and if future generations prefer a higher k, that is less eutrophic or an oligotrophic state, it is very costly for them to restore an oligotrophic state. We found that the higher the phosphorous level at the time is, the larger the value loss for future generations.

We propose that one of the feasible, and theoretically easy ways to solve these problems is to allow a neutral institution to undertake some intervention to bring about a resilient ecosystem with a dominant oligotrophic state, that means the LSS. A resilient ecosystem is defined as a lake which can keep its desirable function to maintain its ability to absorb and biodegrade loading. We know as long as we have a scenario such as LSS, the optimal control will choose the path to the oligotrophic steady state for any given initial phosphorous stock. It should also be mentioned that the scenario of LSS can absorb strong shocks in the sense that it does not exhibit a history dependent property unlike the Skiba scenario. Our strategy is to introduce a tax rate in order to obtain a resilient ecosystem with a dominant oligotrophic state.

In this section, we thus introduce a tax rate on the phosphorous loading whereby the tax rate will depend on the stock of phosphorous. We use a tax rate as a regulatory instrument. We would like to argue that, in terms of equity, a tax on the phosphorous loading might be more reasonable than a compensation from enjoyers to affectors as the Coase solution would suggest. We use the HJB equation again to solve the problem with taxation analytically. The simulation results for this case will be reported in Section 5.

The objective of the regulatory agency becomes:

[2] For more details on the problem of intertemporal justice, see Scholl and Semmler (2002).

$$
\begin{aligned}
W(x,a) &= Max \int_0^{\infty} e^{-rt} U(x,a) dt \\
&= Max \int_0^{\infty} e^{-rt} (u(a,\tau(x)) - kc(x)) dt
\end{aligned} \tag{30}
$$

subject to

$$
\begin{aligned}
\dot{x} = a - \delta x - p(x),\ x(0) = x_0 \\
\text{with} u' > 0,\ u'(0) = +\infty,\ u'' \leq 0 \\
c(0) = 0,\ c' \geq 0,\ c'(0) = 0,\ c'' \geq 0 \\
p(0) = 0,\ p' \geq 0,\ p'(0) = p'(+\infty) = 0 \\
x_m = arg\ max\ p',\ p'' > 0 \text{ for } x < x_m,\ p'' < 0 \text{ for } x > x_m \\
\tau(0) = 0,\ \tau^{'} \geq 0,\ \tau^{'}(0) = 0,\ \tau^{''} \geq 0
\end{aligned}
$$

where $U(x,a) = U_1(a) + U_2(x)$, $U_1(a) = u(a)$, the utility of the affectors; $U_2(x) = -kc(x)$, the utility of the enjoyers, k, the relative importance of affectors and enjoyers.

We presume the same dynamic behavior of the lake as defined before, namely:

$$
\begin{aligned}
\dot{x} = a - \delta x + p(x) \equiv G(x,a) \\
x(0) = x_0
\end{aligned} \tag{31}
$$

where $p(x)$ is a convex-concave function.

We again restrict our analysis by assuming

$$
p(x) < \delta x \text{ for all } x \geq 0 \tag{32}
$$

so that we will avoid inreversibility of the lake state with strictly positive a for all positive x.
To obtain more specific results, we use the following specifications:

$$
u(a,\tau(x)) = 2\{a(1-\tau(x))\}^{\frac{1}{2}};\ 0 \leq \tau(x) < 1 \tag{33}
$$

$$\tau(x) = \gamma x^2 \tag{34}$$

$$c(x) = \frac{1}{2}x^2 \tag{35}$$

$$p(x) = \frac{x^2}{1+x^2}. \tag{36}$$

The HJB-equation for the present model has the form:

$$\begin{aligned} rV(x) &= \max_a [U(x,a) + V'(x)G(x,a)] \\ &= \max_a [u(a) - kc(x) + V'(x)(a - \delta x + p(x))] \end{aligned} \tag{37}$$

We are interested in the effect of proportional taxes on the level of phosphorous. Depending on the parameters, especially on the coefficient, γ, of tax function, it is possible that there will be a change of the type of global dynamics. This then means that the tax rate will enlarge the domain of attraction for which the equilibrium at the low phosphorous level is dominant. We can analyze the effect of tax rates on the global dynamics by computing the value function and detect the dominant equilibrium or the threshold if it occurs. We undertake three steps as outlined before.

Step 1: Compute the steady states for the stationary HJB-equation. If e is an equilibrium then

$$G(e,a) \equiv a - \delta e + p(e) = 0 \tag{38}$$

and

$$rV(e) = u(\delta e - p(e), \tau(e)) - kc(e) \tag{39}$$

$$V'(e) = \frac{u_\tau(\delta e - p(e), \tau(e))\tau'(e) - kc'(e)}{r + \delta - p'(e)} \tag{40}$$

Because the equilibrium e satisfies

$$rV(e) = \max_a [u(a, \tau(e)) - kc(e) + V'(e)(a - \delta e + p(e))], \tag{41}$$

substituting (39) and (40) into (41) yields
$u(\delta e - p(e), \tau(e)) - kc(e)$

$$= \max_a \left[u(a, \tau(e)) - kc(e) + \frac{u_\tau(a, \tau(e))\tau'(e) - kc'(e)}{r + \delta - p'(e)}(a - \delta e + p(e)) \right], \tag{42}$$

Solving $\frac{d}{da}[\cdot] = 0$ gives

$$u_a(a, \tau(e)) + \frac{u_\tau(\delta e - p(e), \tau(e))\tau^{'}(e) - kc^{'}(e)}{r + \delta - p^{'}(e)} = 0 \tag{43}$$

From the specific functions, (33)-(36), we obtain

$$a^{-\frac{1}{2}}(1 - \gamma e)^{\frac{1}{2}} - \frac{2\gamma e\Big(\delta e - \frac{e^2}{1+e^2}\Big)^{\frac{1}{2}}(1 - \gamma e^2)^{-\frac{1}{2}} + ke}{r + \delta - \frac{2e}{(1+e^2)^2}} = 0 \tag{44}$$

or

$$a = \frac{(1 - \gamma e^2)\Big(r + \delta - \frac{2e}{(1+e^2)^2}\Big)^2}{\Big(2\gamma e\Big(\delta e - \frac{e^2}{1+e^2}\Big)^{\frac{1}{2}}(1 - \gamma e^2)^{-\frac{1}{2}} + ke\Big)^2}. \tag{45}$$

Inserting this condition into the steady state condition (38), we have

$$a^{-\frac{1}{2}}(1 - \gamma e^2)^{\frac{1}{2}} - \frac{2\gamma e\Big(\delta e - \frac{e^2}{1+e^2}\Big)^{\frac{1}{2}}(1 - \gamma e^2)^{-\frac{1}{2}} + ke}{r + \delta - \frac{2e}{(1+e^2)^2}} = 0 \tag{46}$$

or

$$\Big(r+\delta-\frac{2e}{(1+e^2)^2}\Big)(1-\gamma e^2)-ke\Big(\delta e-\frac{e^2}{1+e^2}\Big)^{\frac{1}{2}}(1-\gamma e^2)^{\frac{1}{2}}-2\gamma e\Big(\delta e-\frac{e^2}{1+e^2}\Big) = 0 \tag{47}$$

Solving (47) for e, we can obtain multiple steady states. Note that we are only interested in the non-negative and real steady states, $e \geq 0$, for economic reason.

The second-order necessary condition is satisfied by

$$\frac{d^2}{da^2}[\cdot] = u_{aa}(a, \tau(e)) = -\frac{1}{2}a^{-\frac{3}{2}}(1 - \gamma e^2)^{\frac{1}{2}} < 0 \tag{48}$$

since we assume $0 \leq \tau(x) = \gamma e^2 < 1$.

Step 2: Solve the dynamic HJB-equation starting from the equilibrium candidates.

Using the satisfactory HJB-equation again we obtain

$$\begin{aligned} rV(e) &= \max_a [u(a, \tau(e)) - kc(e) + V'(e)(a - \delta e + p(e))] \qquad (49) \\ &= \max_a \left[2(a(1-\gamma e^2))^{\frac{1}{2}} - \frac{1}{2}ke^2 + V'(e)\Big(a - \delta e + \frac{e^2}{1+e^2}\Big) \right]. \end{aligned}$$

Solving $\frac{d}{da}[\cdot] = 0$ gives $a^{-\frac{1}{2}}(1-\gamma e^2)^{\frac{1}{2}} + V'(e) = 0$ or

$$a^{\frac{1}{2}} = -\frac{(1-\gamma e^2)^{\frac{1}{2}}}{V'(e)} \text{ and } a = \frac{1-\gamma e^2}{V'(e)^2}. \qquad (50)$$

Substituting (50) into (49) gives

$$rV(e) = -\frac{1-\gamma e^2}{V'(e)} - \frac{1}{2}ke^2 - V'(e)\Big(\delta e - \frac{e^2}{1+e^2}\Big), \qquad (51)$$

therefore

$$V'(e)^2 + \frac{rV(e) + kc(e)}{\delta e - p(e)}V'(e) + \frac{1-\gamma e^2}{\delta e - p(e)} = 0. \qquad (52)$$

Then, by solving (52) for $V'(e)$, we obtain an ordinary differential equation in V with candidates of steady states as initial conditions:

$$V'(e) = -\frac{rV(e) + kc(e)}{2(\delta e - p(e))} \pm \sqrt{\Big(\frac{rV(e) + kc(e)}{2(\delta e - p(e))}\Big)^2 - \frac{1-\gamma e^2}{\delta e - p(e)}}. \qquad (53)$$

Using this information for the solution of V we get

$$V'(x) = -\frac{rV(x) + kc(x)}{2(\delta x - p(x))} - \sqrt{\Big(\frac{rV(x) + kc(x)}{2(\delta x - p(x))}\Big)^2 - \frac{1-\gamma x^2}{\delta x - p(x)}} \text{ for } x \geq e. \qquad (54)$$

and

$$V'(x) = -\frac{rV(x) + kc(x)}{2(\delta x - p(x))} + \sqrt{\left(\frac{rV(x) + kc(x)}{2(\delta x - p(x))}\right)^2 - \frac{1 - \gamma x^2}{\delta x - p(x)}} \text{ for } x < e. \tag{55}$$

with candidates of steady states from (47) as initial conditions. Note that, from the assumption (32), $p(x) < \delta x$, $V'(x) < 0$ for any x since the stock of phosphorous, x, impacts negatively the value function.

Step 3: Solve the global value function.

We can compute the global value function for the original problem by

$$V(x) = maxV_i(x). \tag{56}$$

5 Numerical Results using Dynamic Programming

5.1 The Study of the Equilibria

Using the aforementioned method, in particular Step 1, we employ numerical examples to carry out the study of equilibrium candidates for the models for both with and without taxation.

Numerical Examples without Taxation

In the model without taxation, we vary the relative importance between affectors and enjoyers, k. k takes a number between 0 and 1 and k is lower when enjoyers of the lake do not care much about the quality of the lake. We have always three equilibria, except in case 1 where there is a unique equilibrium. Figures for this section are collected in the appendix.

Case 1: There is a unique equilibrium at the low level of phosphorous. δ= 0.55, r= 0.1, k= 1, e= 0.344464

Case 2: LSS; there are three equilibria and the low equilibrium is dominant. δ=0.55, r= 0.1, k= .75, e_1= 0.344464 (Phase 1)

Case 3: Skiba; there are three equilibria and the Skiba point exists between the low and high equilibria. δ= 0.55, r= 0.1, k= .5, e_1= 0.397716, e_2= 0.893286, e_3= 1.86141 (Phase 2)

Case 4: USS; there are three equilibria and the high equilibrium is dominant. δ= 0.55, r= 0.1, k= .3, e_1= 0.431442, e_2= 0.795756, e_3= 2.5487 (Phase 3)

Numerical Examples with Taxation

We now apply a tax system where the loading of phosphorous is taxable but the tax rate is state-dependent. Our simulations focus on the change of global dynamics by varying the coefficient of the tax rate, γ. We start from the situation of case 4, in the model without taxation, that is $\gamma = 0$ where there are three equilibria and the high equilibrium is dominant (USS, Phase 3). The purpose of this taxation is, raising γ gradually, to verify whether this tax system work for recovering a resilient ecosystem with dominant oligotrophic state (USS).

Case 5: USS; there are three equilibria and the high equilibrium is still dominant. $\delta = 0.55,\ r = 0.1,\ k = .3,\ \gamma = 0.1,\ e_1 = 0.419857,\ e_2 = 0.823482,\ e_3 = 1.82848$ (Phase 4)

Case 6: Skiba; there are three equilibria and the Skiba point exists between the low and high equilibria. $\delta = 0.55,\ r = 0.1,\ k = .3,\ \gamma = 0.3,\ e_1 = 0.399431,\ e_2 = 0.902914,\ e_3 = 1.28307$ (Phase 5)

Case 7: LSS; there are three equilibria and the low equilibrium is dominant. $\delta = 0.55,\ r = 0.1,\ k = .3,\ \gamma = 0.37,\ e_1 = 0.39299,\ e_2 = 0.96092,\ e_3 = 1.14098$ (Phase 6)

Case 8: There is a unique equilibrium at the low level of phosphorous. $\delta = 0.55,\ r = 0.1,\ k = .3,\ \gamma = 0.5,\ e = 0.381835$. $\delta = 0.55,\ r = 0.1,\ k = .3,\ \gamma = 1,\ e = 0.346451$

Next do now investigate the global dynamics of several of these cases in more detail using a numerical solution of the HJB equation (10) by means of the dynamic programming approach. This algorithm, which goes back to Capuzzo–Dolcetta and Falcone (see Appendix 1 of Bardi and Capuzzo Dolcetta (1997)), allows the approximation of the optimal value functions and optimal controls in feedback form by a suitable discretization of the problem first in time and then in space. While in principle the version of the algorithm used here allows an adaptive grid discretization, for the one dimensional problem at hand it turned out that equidistant grids provide enough accuracy and hence adaptive grids were not necessary here.

5.2 The Model without Taxation

Our first set of numerical computations uses the basic model without taxation. We fix the parameter values $\alpha = 2$, $\sigma = \beta = 0.5$, $m = n = 1$, $\rho = 2$, $\delta = 0.55$ and $r = 0.1$. For these parameters, depending on k, different scenarios for the global behavior of the optimal trajectories can be obtained: for small k there exists just one optimal equilibrium, while for increasing k another equilibrium appears to the left of the first, where the domains of attraction of the two equilibria is divided by a Skiba point (the optimal control is discontinuous at this point). Increasing k even more, the first (right) equilibrium and the Skiba point vanish and just one equilibrium (the left one) remains. Table 1 gives the computed values for these equilibria for different values of k.

k	left eq.	Skiba point	right eq.
0.3	—	—	2.548
0.5	0.396	0.714	1.861
0.7	0.376	1.154	1.444
0.75	0.366	—	—
1	0.346	—	—

Table 1: Equilibria depending on k

The Figures 1 and 2 show the optimal value functions and optimal control laws in feedback form for $k = 0.3$, 0.5 and 0.75. Note that the Skiba point at $x_S = 0.714$ for $k = 0.5$ is clearly visible as a kink in the optimal value function and as a jump in the optimal control law.

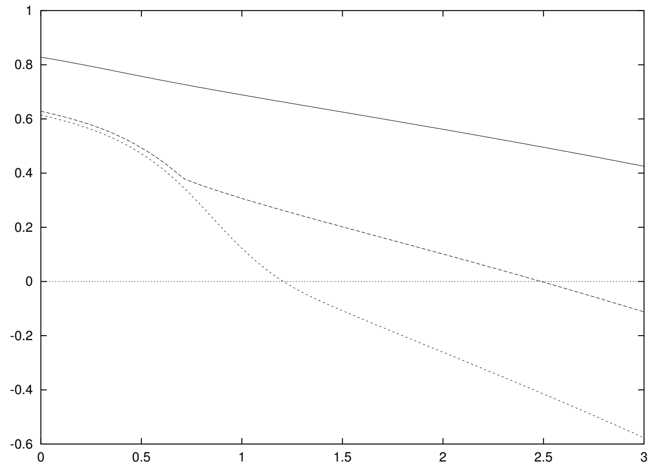


Figure 1: Optimal value functions for $k = 0.3,\ 0.5,\ 0.75$ (top to bottom)

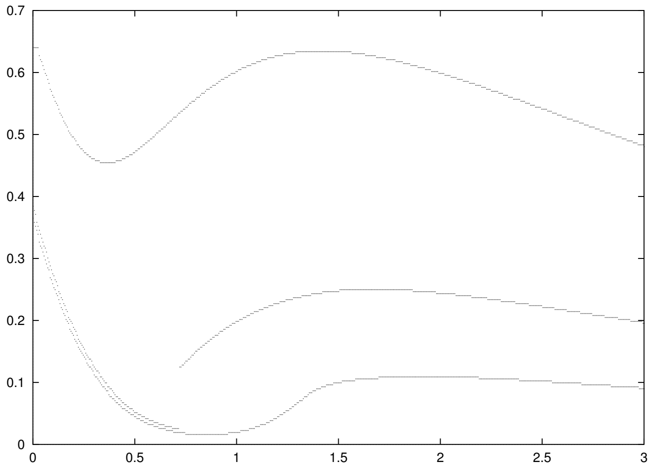


Figure 2: Optimal controls for $k = 0.3,\ 0.5,\ 0.75$ (top to bottom)

The numerical computations were performed with time discretization step $h = 1/20$ and equidistant grid with space discretizations space $k = 1/200$. The set of control values was discretized with 201 equidistributed values.

5.3 The Model with Taxation

We have repeated our computations for the model with taxation using the same parameters as above and for $k = 0.3$, i.e., for the situation where only one optimal equilibrium with relatively high pollution of the lake is observed in the numerical simulation. Here we investigate the effect of different levels of tax γ on the dynamical behavior of the optimally controlled system. For small γ the situation remains basically the same as for the model without tax: we observe just one optimal equilibrium with rather high pollution. For increasing values of γ the situation is similar as for increasing values of k, above, i.e., first a second equilibrium to the left of the first appears (with domains of attraction separated by a Skiba point) and then the first equilibrium (and the Skiba point) vanish. Table 2 shows the numerical values for these points for different tax levels γ. In addition, the Figures 3 and 4 show the optimal value functions and optimal control laws for selected values of γ.

γ	left eq.	Skiba point	right eq.
0	—	—	2.548
0.1	—	—	1.831
0.2	0.411	0.95	1.502
0.3	0.401	0.75	1.281
0.37	0.393	—	—
0.4	0.391	—	—
0.5	0.381	—	—
1	0.346	—	—

Table 2: Equilibria depending on γ

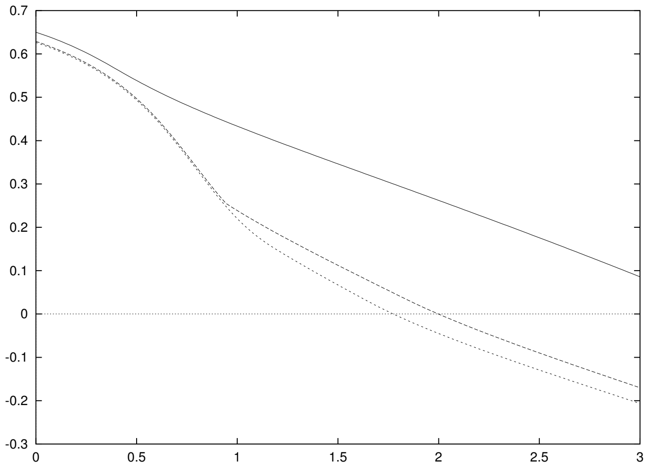


Figure 3: Optimal value functions for $\gamma = 0.1,\, 0.3,\, 0.37$ (top to bottom)

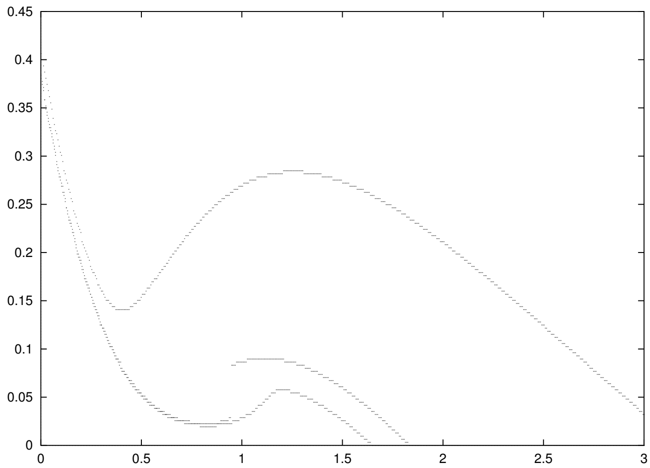


Figure 4: Optimal controls for $\gamma = 0.1,\, 0.3,\, 0.37$ (top to bottom)

It should be noted that – even though the value function for the model without tax is, of course, larger – its value in the optimal equilibrium is larger in the case with tax: for $\gamma = 0$ the optimal equilibrium is $x^* = 2.548$ with $V(x^*) = 4.90$, while for $\gamma = 0.5$ the optimal equilibrium is 0.381 with $V(x^*) = 5.32$. This comes from the fact that the equilibria switch around with changing tax rates. Yet, overall we can observe in Figure 3, that that the value function is always lower with higher tax rates. Moreover, the jump of the control variable at the Skiba point is also clearly visible in Figure 4 for the tax rate $\gamma = 0.3$.

6 Conclusions

Using the Hamilton-Jacobi-Bellmann equation, we have studied an ecological management problem with multiple equilibria and different scenarios for the global dynamics. When such different scenarios in the multiple equilibria model arise, knowing the global value function is crucial to detext the unique optimal path and the global policy function. The global value function is used for path selection for given initial state. One path may generate a higher value than the the other paths for any given initial state. Yet, those best paths may lead to low or high pollution attractors or threshold may exist between two attractors and the optimal path will switch at the threshold – the Skiba point.

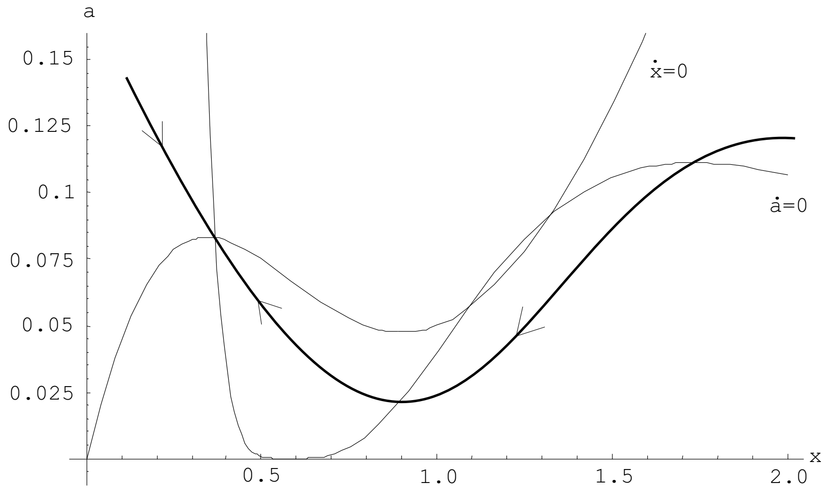
For our model of a shallow lake we have explored the possibility of enlarging a resilient domain of an oligotrophic attractor. Introducing a state dependent tax rate on the phosphorous loading, we demonstrated that the goal of maintaining or enhancing a desirable resilient system could be achieved with state dependent tax rates. Such an intentional action by some regulatory agency should be justified since there exist significant costs to being trapped into an eutrophic state. There are negative externalities from the lake being locked into a eutrophic state. Those concerns not only immediate effects on the ecosystem but also future generations environmental conditions. Those externalities were not explicitly theoretically treated in order to avoid greater complexity.

However, we can have an intuition, from or numerical example, Figure 1, on how the future generation is affected by the current generation's choice. Figure 1 shows the value functions for different $k's$, the preferences for the lake state over the benefit from phosphorous loading. It is clear that the value loss from increasing k is larger for the higher level of phosphorous stock, x. It means that the future generation favoring cleaner lake will suffer a large loss of value by attaining the oligotrophic state when the eutrophic state is already given as an initial state. Therefore, maintaining the resilient system with oligotrophic attractor might be a safe way to avoid the unfair treatment of different generations. It is also remarkable that the system is strongly resilient in the face of shocks, preventing a lake from flipping over easily, unlike the Skiba case which has the property of history dependence.

We use taxation as an instrument of a regulatory intervention. It seems to work well for the purpose of restoring a resilient ecosystem. As a result, however, there exists a value loss after increasing the coefficient of the tax rate, γ. We did not compare tax systems or environmental policies from the point of view of value loss. Indeed, the value function can be utilized for policy evaluation.

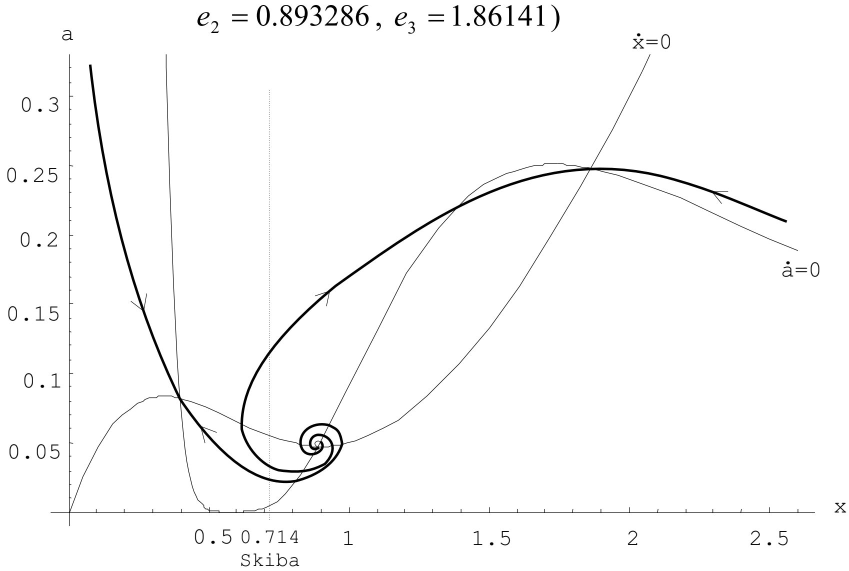
Phase 1

(0.55 , r 0.1 , k .75 , e_1 0.36753, e_2 1.08782 , e_3 1.3262)



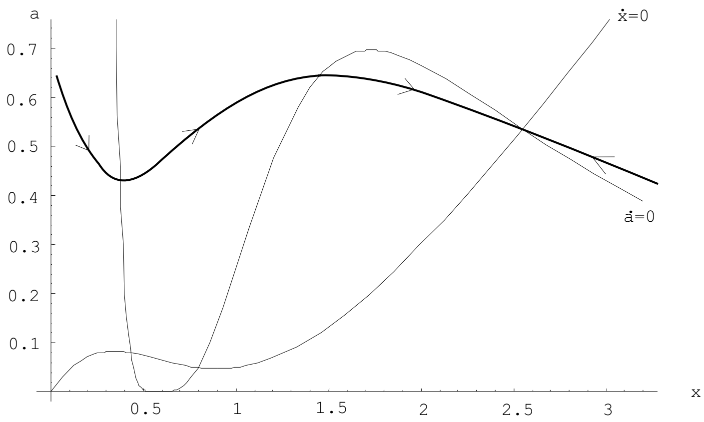
Phase 2

$(\delta = 0.55$, $r = 0.1$, $k = 0.5$, $e_1 = 0.397716$, $s = 0.714$,
$e_2 = 0.893286$, $e_3 = 1.86141)$



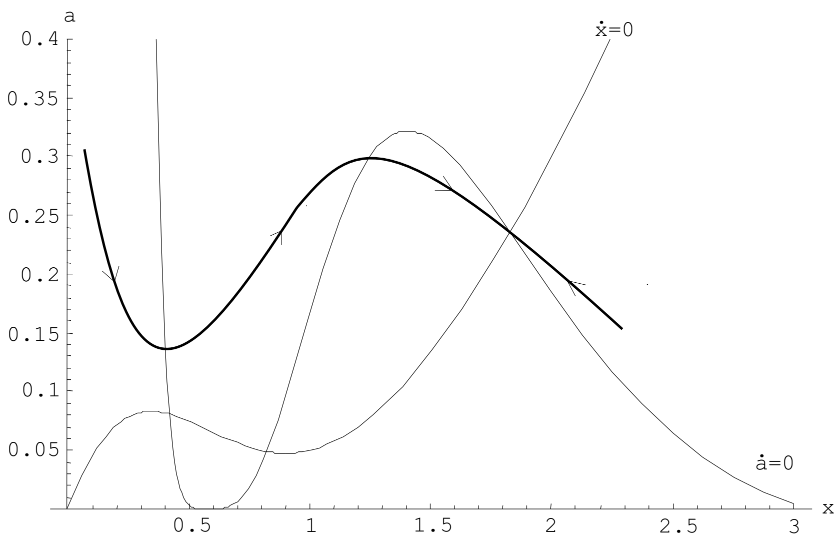
Phase 3

($\delta = 0.55$, $r = 0.1$, $k = 0.3$, $e_1 = 0.431442$, $e_2 = 0.795756$, $e_3 = 2.54847$)



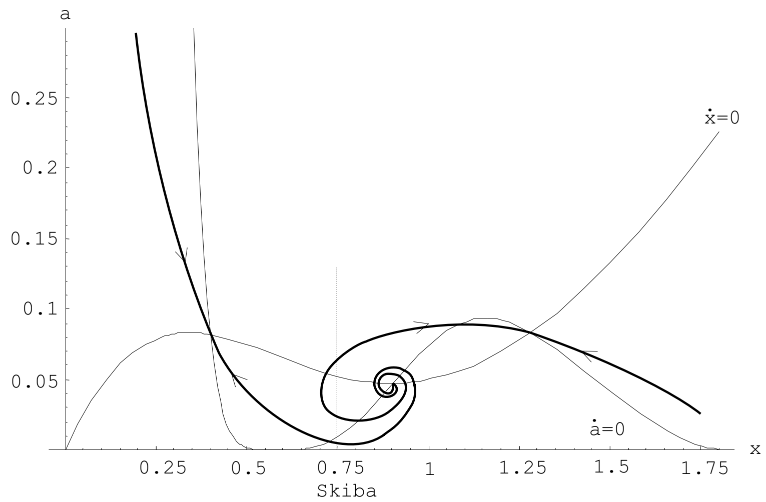
Phase. 4

($\delta = 0.55$, $r = 0.1$, $k = 0.3$, $\gamma = 0.1$, $e_1 = 0.419857$,
$e_2 = 0.823482$, $e_3 = 1.82848$)



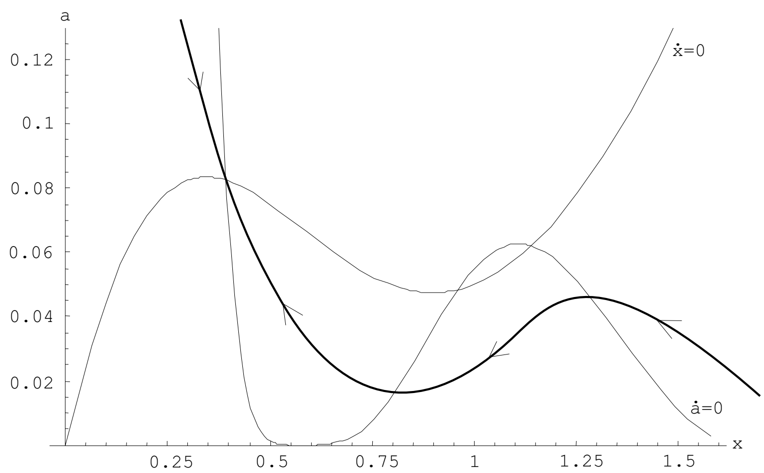
Phase 5

($\delta = 0.55$, $r = 0.1$, $k = 0.3$, $\gamma = 0.3$, $e_1 = 0.399431$, $s = 0.75$, $e_2 = 0.902914$, $e_3 = 1.28307$)



Phase 6

($\delta = 0.55$, $r = 0.1$, $k = 0.3$, $\gamma = 0.37$, $e_1 = 0.39299$, $e_2 = 0.96092$, $e_3 = 1.14098$)



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