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On the dynamics of an endogenous growth model with learning by doing

by

Alfred Greiner

University of Bielefeld
Department of Economics
Center for Empirical Macroeconomics
P.O. Box 100 131

33501 Bielefeld, Germany

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Alfred Greiner†

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Abstract

The paper studies the local dynamics of an endogenous growth model with externalities of investment. It is demonstrated that, in case of sustained per capita growth, the competitive economy is characterized by a situation with a unique balanced growth path which is saddle point stable or by a situation with two balanced growth paths. If there are two balanced growth paths, the one with the higher growth rate is a saddle point whereas the path with the lower growth rate is either completely stable, with convergence to a rest point or limit cycle, or completely unstable. In the social optimum the existence of a balanced growth path implies that it is unique and that this path is a saddle point.

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†Department of Economics, University of Bielefeld, P.O. Box 100131, 33501 Bielefeld, Germany, e-mail: agreiner@wiwi.uni-bielefeld.de

1 Introduction

One strand in endogenous growth theory is concerned with sustained per capita growth resulting from positive externalities of investment in physical capital. That idea goes back to Arrow (1962) who asserted that learning by doing is an important determinant of the stock of knowledge of workers in an economy. A good index for the stock of knowledge, according to Arrow, is cumulated investment. Levhari (1966) generalized the model presented by Arrow, who originally used a vintage approach, and showed that most of Arrow's results can be extended to any homogenous production function of the first degree which is not restricted to fixed coefficients.

Sheshinski (1967) and Romer (1986) have integrated that idea in the neoclassical growth model with optimizing agents and Romer could demonstrate that his model may generate sustained per capita growth with an endogenously determined growth rate. However, in contrast to other endogenous growth models, especially models of the Lucas-Uzawa type,[1] the growth model with external effects of investment has not been anaylzed frequently as to its dynamics. Benhabib and Farmer (1994) for example have demonstrated that an endogenous growth model with externalities may generate indeterminacy in the competitive economy if labour supply is elastic. But, as far as I know, there does not exist a thorough dynamic analysis of the basic endogenous growth model with inelastic labour supply, which is more general concerning the formation of knowledge capital as a by-product of investment.

Since empirical studies suggest that investment is associated with positive externalities (see e.g. DeLong and Summers (1991) or Hamilton and Monteagudo (1998)) it seems necessary to integrate that feature in a growth model and to explicitly study its dynamics.

In the rest of the paper we proceed as follows. In section 2 we present the growth model with external effects of investment and derive optimality conditions for the competitive economy and for the social optimum. Section 3 studies the dynamics of the model for the two economies and section 4 finally concludes the paper.

[1]See e.g. the papers by Xie (1994) and Benhabib and Perli (1994).

2 The model

Our economy is represented by one household whose goal is to maximize its discounted stream of utilities arising from consumption subject to a budget constraint. Further, there is an externality associated with investment in physical capital which consists in building up a stock of knowledge capital (learning by doing). We start with the description of that external effect.

The external effect of investment

We assume that the stock of knowledge capital, $A(t)$, which raises the efficiency of labour input, is formed as a by-product of gross investment in physical capital according to $A(t) = \varphi \int_{-\infty}^{t} e^{\eta(s-t)} I(s) ds$. The use of the weighting function $e^{\eta(s-t)}$ implies that investment further back in time contributes less to the current stock of knowledge capital than more recent gross investment. As to the use of weighting functions in economics see e.g. Ryder and Heal (1973), Feichtinger and Sorger (1988) or Grossman and Helpman (1991), chap. 3.2. Differentiating $A(t)$ with respect to time[2] leads to

$$\dot{A} = \varphi I - \eta A, \; A(0) = A_0. \tag{1}$$

That equation shows that $\varphi > 0$ states how much any unit of investment contributes to the formation of the stock of knowledge and $\eta \geq 0$ gives the depreciation of knowledge capital. Depreciation of knowledge can be justified by supposing that any new capital good requires new knowledge in order to be operated efficiently.

Next, we describe the competitive economy and the social optimum.

The Competitive Economy

As mentioned above, our economy consists of a representative household whose goal is to maximize the discounted stream of utility over an infinite time horizon:

$$\max_{\{C(t)\}} \int_0^\infty e^{-\rho t} u(C(t)) dt, \tag{2}$$

[2]In the following we suppress the time argument if no ambiguity results.

subject to the budget constraint[3]

$$\dot{K} = A^{\alpha} K^{1-\alpha} - C - \delta K, \quad K(0) = K_0, \tag{3}$$

where $u(\cdot)$ is a strictly concave utility function, $u'(\cdot) > 0$, $u''(\cdot) < 0$. ρ denotes the constant rate of time preference, K the stock of physical capital, which depreciates at the rate $\delta \geq 0$. The labor supply is assumed to be constant and normalized to one so that all variables denote per capita quantities. $\alpha \in (0,1)$ gives the labour share in the Cobb-Douglas production function and $1-\alpha$ is the share of physical capital. It should be noted that the representative household has rational expectations, which is equal to perfect foresight in a deterministic setup, and knows the evolution of $\{A(t),\ t \in [0,\infty)\}$. But he does not take it into account in solving his optimization problem. That is, as usual, the external effect of investment is not taken into account in the competitive economy.

To obtain optimality conditions we formulate the current-value Hamiltonian $H = u(C) + \lambda(A^{\alpha} K^{1-\alpha} - C - \delta K)$. The necessary optimality conditions are given by

$$u'(C) = \lambda, \tag{4}$$

$$\dot{\lambda} = (\rho + \delta)\lambda - \lambda(1-\alpha)A^{\alpha} K^{-\alpha}, \tag{5}$$

The necessary optimality conditions are also sufficient if the limiting transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda(t)K(t) \geq 0$ is fulfilled.

The social optimum

Since there is a positive externality associated with investment in physical capital it is obvious that the optimization problem of the competitive economy does not yield the social optimum. The latter is obtained by solving the optimization problem

$$\max_{\{C(t)\}} \int_0^{\infty} e^{-\rho t} u(C(t))dt, \tag{6}$$

[3]The model with household production is equivalent to a decentralized economy where the household receives labour income and income from its saving, and where the wage and interest rate are equal to the marginal product of labour and capital respectively.

subject to

$$\dot{K} = A^{\alpha}K^{1-\alpha} - C - \delta K, \quad K(0) = K_0, \tag{7}$$

$$\dot{A} = \varphi(A^{\alpha}K^{1-\alpha} - C) - \eta A, \quad A(0) = A_0. \tag{8}$$

At that point it should also be mentioned that, from a formal point of view, our framework is equal to the model presented by Romer (1986) if we set $\varphi = 1$ and $\delta = \eta = 0$. Then, knowledge and physical capital evolve at the same pace such that those two state variables can be merged into one single variable. But it must be underlined that Romer has increasing returns in the factors which can be accumulated while our economy is characterized by a constant returns to scale technology. The dynamics of the Romer model have been studied heuristically in Xie (1991).

To find necessary conditions we formulate the current-value Hamiltonian $H = u(C) + \lambda_1(A^{\alpha}K^{1-\alpha} - C - \delta K) + \lambda_2(\varphi(A^{\alpha}K^{1-\alpha} - C) - \eta A)$. Necessary conditions are then obtained as

$$u'(C) = \lambda_1 + \lambda_2\varphi, \tag{9}$$

$$\dot{\lambda}_1 = (\rho + \delta)\lambda_1 - \lambda_1(1-\alpha)K^{-\alpha}A^{\alpha} - \lambda_2\varphi(1-\alpha)K^{-\alpha}A^{\alpha}, \tag{10}$$

$$\dot{\lambda}_2 = (\rho + \eta)\lambda_2 - \lambda_1\alpha A^{\alpha-1}K^{1-\alpha} - \lambda_2\varphi\alpha A^{\alpha-1}K^{1-\alpha}. \tag{11}$$

Again, the necessary conditions are sufficient if the limiting transversality condition $\lim_{t\to\infty} e^{-\rho t}(\lambda_1(t)K(t) + \lambda_2(t)A(t)) \geq 0$ is fulfilled.

From (4) and (9) it is immediately seen that in the social optimum the level of investment is higher. That holds because in the social optimum investment is not only paid its shadow price λ_1 but also an additional weighted shadow price $\varphi\lambda_2$. Consequently, as usual, the government has to give incentives for investment by, for example, raising a lump-sum tax which is used to subsidize investment. It should be noted that the higher φ, i.e. the more any unit of investment contributes to the growth of the stock of knowledge, the higher the subsidy has to be.

In the next section we study the dynamics of that model around a balanced growth path (BGP) both for the competitive economy and for the social optimum.

3 The dynamics

To investigate the dynamics of our model we first derive the equations of motions.

For the competitive economy the differential equation system describing our economy is obtained by differentiating (4) with respect to time and using (5). Taking into account that the stock of knowledge evolves according to (1) and physical capital according to (3) leads to

$$\begin{aligned} \dot{C} &= A^{\alpha}K^{-\alpha}C\left(\frac{1-\alpha}{\sigma}\right) - C\left(\frac{\rho+\delta}{\sigma}\right), && (12) \\ \dot{K} &= A^{\alpha}K^{1-\alpha} - C - \delta K, && (13) \\ \dot{A} &= \varphi A^{\alpha}K^{1-\alpha} - \varphi C - \eta A, && (14) \end{aligned}$$

with $\sigma \equiv -u''(C)C/u'(C)$ the inverse of the intertemporal elasticity of substitution which is assumed to be constant.

We are interested in a BGP on which all variables grow at the same constant growth rate, i.e. a path on which $\dot{C}/C = \dot{K}/K = \dot{A}/A \equiv g_c$ holds. To gain further insight in such a path we introduce $c = C/A$ and $k = K/A$. Differentiating those ratios with respect to time gives $\dot{c}/c = \dot{C}/C - \dot{A}/A$ and $\dot{k}/k = \dot{K}/K - \dot{A}/A$ or explicitly:

$$\begin{aligned} \frac{\dot{k}}{k} &= -\delta - \frac{c}{k} + \eta + \varphi c + (1-\varphi k)k^{-\alpha}, && (15) \\ \frac{\dot{c}}{c} &= -\frac{\rho+\delta}{\sigma} + \frac{1-\alpha}{\sigma}k^{-\alpha} + \eta + c\varphi - \varphi k^{1-\alpha}. && (16) \end{aligned}$$

A rest point of system (15)-(16) corresponds to a BGP of (12)-(14) with $\dot{A}/A = \dot{K}/K = \dot{C}/C = g_c = const.$

For the social optimum a BGP is defined as a path on which $\dot{A}/A = \dot{K}/K = (-1/\sigma)(\dot{\lambda}_1/\lambda_1) = (-1/\sigma)(\dot{\lambda}_2/\lambda_2) = g_s = const.$ holds. It should be noted that $(-1/\sigma)(\dot{\lambda}_1/\lambda_1) = (-1/\sigma)(\dot{\lambda}_2/\lambda_2) = g_s$ implies $\dot{C}/C = g_s$. That can easily be seen if we differentiate (9) with respect to time giving

$$\frac{\dot{C}}{C} = -\frac{1}{\sigma}\left(\frac{\dot{\lambda}_1 + \varphi\dot{\lambda}_2}{\lambda_1 + \varphi\lambda_2}\right). \quad (17)$$

The system of differential equations describing the dynamics around a BGP is given by $\dot{c}/c = \dot{C}/C - \dot{A}/A$ and $\dot{k}/k = \dot{K}/K - \dot{A}/A$ which is

$$\frac{\dot{k}}{k} = -\delta - \frac{c}{k} + \eta + \varphi c + (1 - \varphi k)k^{-\alpha}, \tag{18}$$

$$\frac{\dot{c}}{c} = -\frac{1}{\sigma}\frac{\dot{\lambda}_1 + \varphi\dot{\lambda}_2}{\lambda_1 + \varphi\lambda_2} + \eta + c\varphi - \varphi k^{1-\alpha}. \tag{19}$$

It should be noted that we confine our analysis to interior BGPs only, i.e. BGPs with $\bar{k} \neq 0$ and $\bar{c} \neq 0$, so that we can consider system (15)-(16) and (18)-(19) in the rates of growth.[4] We do that because $\bar{k} = 0$ is not feasible since it is raised to a negative power in (16) and (19). Further, $\bar{c} = 0$ would imply that the level of consumption is zero which does not make sense from the economic point of view.

Next, we study the dynamics of our growth model for the case of sustained per-capita growth. First we consider the competitive economy.

The competitive economy

In the competitive economy we can observe either a unique BGP which is a saddle point or two BGPs where the one giving the higher balanced growth rate is a saddle point whereas the BGP yielding the lower growth rate cannot be a saddle point. Proposition 1 gives the result.

Proposition 1 *For the competitive economy the following holds:*
(i) If $\delta \geq \eta$ the existence of a BGP implies that it is unique and the BGP is stable in the saddle point sense.
(ii) If $\delta < \eta$ the following is true: If $(\rho + \delta)/\sigma \leq \delta$ the existence of a BGP implies that it is unique and that it is a saddle point. If $(\rho + \delta)/\sigma > \delta$ there exist two BGPs in case of sustained per capita growth or a unique BGP which, however, is not generic. The BGP giving the higher growth rate is a saddle point, the BGP yielding the lower growth rate cannot be a saddle point.

Proof: See appendix.

[4] The bar $^-$ denotes the values for k and c on the BGP.

That proposition tells us that the competitive economy may be both globally and locally indeterminate, around the BGP with the lower growth rate (as to the economic interpretation of local and global indeterminacy see e.g. Benhabib and Framer (1994)).

From the technical point of view, local indeterminacy can be observed if the parameter constellation is such that the trace of the Jacobian matrix is smaller zero so that both eigenvalues have negative real parts. The calculation of the trace of the Jacobian in our model is straightforward and we can give conditions which must be fulfilled so that the trace is negative. However, those conditions are purely technical and cannot be interpreted from an economic point of view.[5]

As to the plausibility of the parameters necessary for indeterminacy we can say the following. First, $\sigma < 1 + \rho/\delta$ means a relatively high intertemporal elasticity of substitution. Most of the empirical work estimating σ gives values at or above unity (see e.g. Blanchard and Fischer, 1989, p. 44) although there are even some estimates which obtain lower values (see e.g. Boskin, 1978, or Amano and Wirjanto, 1998). Those estimations demonstrate that this necessary condition for indeterminacy is in line with empirical observations. Second, the requirement $\delta < \eta$ states that the depreciation of physical capital must be smaller than that of knowledge capital. I do not know of empirical studies trying to find the depreciation rate of knowledge capital. However, given the fact that technical change accelerates in Western economies it is to be expected that existing knowledge becomes more and more rapidly obsolete so that this inequality may be fulfilled. In the next subsection we consider the dynamics of the social optimum.

The social optimum

The social optimum is described by (18)-(19). To study the local dynamics of that system in the neighborhood of a BGP we linearize (18)-(19) at the rest point $(\bar{k}, \bar{c})$, where (18)-(19) is a function of the variables k and c, and of constant parameters,

[5] Using Hopf bifurcation theory it has been demonstrated in Greiner and Semmler (1996) that convergence to a limit cycles may occur in this model, too.

alone. It turns out that the social optimum is always characterized by a unique BGP which is a saddle point. That is the contents of proposition 2.

Proposition 2 *For the social optimum the existence of a BGP implies that it is unique and that it is saddle point stable.*

Proof: Analogous to that of proposition 1. It is available on request.

4 Conclusion

This paper has given a characterization of a generalized version of the endogenous growth model by Romer (1986) with constant returns to scale, without resorting to numerical simulations. This is of importance because this type of growth models is of high empirical relevance but there does not yet exist a thorough characterization of its dynamics. The results demonstrate that the social optimum is always both globally and locally determinate. This is not too surprising since the social optimum is the solution of a concave optimization problem. The competitive economy, however, may give rise to global and local indeterminacy. A necessary condition for that outcome is a relatively high intertemporal elasticity of substitution which, however, is also compatible with empirical estimations. Thus, the basic endogenous growth model with externalities of investment can explain why countries may converge to different growth paths in the long run.

Another aspect we want to point out is our modellation of the external effect of investment. So, in the competitive economy the positive externalities do not only lead to a higher share of (physical) capital in the aggregate production function but they give rise to an additional differential equation which affects the dynamics to a great degree. This can also be applied to models with other types of externalities, like environmental pollution for example, so that such model are likely to have a more complex dynamic behavior than convergence to a unique rest point.

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Appendix

Proof of proposition 1: Part (i) with $\delta > \eta$. (15)=0 yields $\bar{c}$. Inserting $\bar{c}$ in $\dot{c}/c$ yields[6]

$$f(k,\cdot) = -\frac{\rho+\delta}{\sigma} + \eta + \frac{1-\alpha}{\sigma}k^{-\alpha} + (\eta-\delta)\frac{\varphi k}{1-\varphi k}\,.$$

$f(k,\cdot)$ is continuous for $k \in (0,\varphi^{-1})$ and $\lim_{k\to 0} f(k,\cdot) = +\infty$, $\lim_{k\nearrow\varphi^{-1}} f(k,\cdot) = -\infty$ ($\nearrow$ means that k approaches φ^{-1} from below). Further,

$$\frac{\partial f(k,\cdot)}{\partial k} = \frac{\varphi(\eta-\delta)}{(1-k\varphi)^2} - \frac{\alpha(1-\alpha)}{\sigma}k^{-\alpha-1} < 0, \text{ for } \delta > \eta.$$

[6] $\delta > \eta$ requires $\varphi k < 1$ for $C < K^{1-\alpha}A^{\alpha}$.

Stability. $\det J < 0$ is necessary and sufficient for saddle point stability. $\det J$ is

$$\det J = (1-\bar{k}\varphi)\bar{k}^{-1}\left(\frac{\varphi(\eta-\delta)}{(1-\bar{k}\varphi)^2} - \frac{\alpha(1-\alpha)}{\sigma}\bar{k}^{-\alpha-1}\right).$$

$\delta > \eta$, $1-\varphi\bar{k} > 0$ implies $\det J < 0$.

If $\delta = \eta$, $\bar{k} = 1/\varphi$. $\bar{k} = 1/\varphi$ in $\dot{c}/c = 0$ gives $\bar{c}$. $\bar{k}$ and $\bar{c}$ in the Jacobian gives the eigenvalues as $Ev_1 = \varphi$ and $Ev_2 = \varphi(-\eta + (\delta+\rho+(\alpha-1)\varphi^{\alpha})/\sigma)$. For $\bar{c} < \bar{k}^{1-\alpha}$, Ev_2 is always negative. Thus, part (i) is proved.

Part (ii). A k which solves $f(k,\cdot) = 0$ also solves $f(k,\cdot)k^{\alpha}(1-\varphi k) \equiv f_1(k,\cdot) = 0$, with $k^{\alpha}(1-\varphi k) \neq 0$, and vice versa. Therefore, we can analyze the function $f_1(k,\cdot)$ instead of $f(k,\cdot)$.

Multiplying $f(k,\cdot)$ with $k^{\alpha}(1-\varphi k) \neq 0$ leads to

$$f_1(k,\cdot) = \left(\frac{(1-\alpha)}{\sigma}\right)(1-\varphi k) + k^{\alpha}\left(\eta - \frac{\rho+\delta}{\sigma}\right) + k^{1+\alpha}\varphi\left(-\delta + \frac{\rho+\delta}{\sigma}\right),$$

which is continuous for all $k \in (0,\infty)$. Further, $\lim_{k\to 0} f_1(k,\cdot) = (1-\alpha)/\sigma > 0$ and

$$\frac{\partial f_1(k,\cdot)}{\partial k} = -\left(\frac{(1-\alpha)}{\sigma}\right)\varphi + \alpha k^{\alpha-1}\left(\eta - \frac{\rho+\delta}{\sigma}\right) + (\alpha+1)k^{\alpha}\varphi\left(-\delta + \frac{\rho+\delta}{\sigma}\right).$$

4 cases must be distinguished:

1. $(\delta+\rho)/\sigma \le \delta$, $\eta \le (\delta+\rho)/\sigma$ implies $\lim_{k\to\infty} f_1(k,\cdot) = -\infty$ and $\partial f_1(k,\cdot)/\partial k < 0$ for all $k > 0$ so that for this case uniqueness is immediately seen.

2. $(\delta+\rho)/\sigma \le \delta$, $\eta > (\delta+\rho)/\sigma$. Existence of a BGP implies $\partial f_1(k,\cdot)/\partial k < 0$ at least locally. Since $\partial^2 f_1(k,\cdot)/\partial k^2 < 0$ holds for all $k > 0$, $\partial f_1(k,\cdot)/\partial k > 0$ is not feasible once $\partial f_1(k,\cdot)/\partial k$ has become negative and, consequently, no second BGP.

3. $(\delta+\rho)/\sigma > \delta$, $\eta \le (\delta+\rho)/\sigma$. At the first BGP $\partial f_1(k,\cdot)/\partial k < 0$ must hold. At the second BGP $\partial f_1(k,\cdot)/\partial k > 0$ must hold. Since $\lim_{k\to\infty}(\partial f_1(k,\cdot)/\partial k) = \lim_{k\to\infty} f_1(k,\cdot) = \infty$ and $\partial^2 f_1(k,\cdot)/\partial k^2 > 0$ holds for all $k > 0$, there exists a finite k such that $f_1(k,\cdot) = 0$ holds and a second BGP exists. No third BGP exists since $\partial^2 f_1(k,\cdot)/\partial k^2 > 0$.

4. $(\delta+\rho)/\sigma > \delta$, $\eta > (\delta+\rho)/\sigma$. $\lim_{k\to 0}(\partial f_1(k,\cdot)/\partial k) = \infty$ and $\lim_{k\to\infty} f_1(k,\cdot) = \lim_{k\to\infty}(\partial f_1(k,\cdot)/\partial k) = \infty$. Since there is a unique inflection point of $f_1(k,\cdot)$ given

by $k_w = (1-\alpha)(\eta - (\rho+\delta)/\sigma)/(\varphi(1+\alpha)(-\delta + (\rho+\delta)/\sigma)) > 0$ only two BGPs can exist. k such that $f_1(k,\cdot) = 0$, $\partial f_1(k,\cdot)/\partial k = 0$ implies a unique BGP, but the set $\{k : f_1(k,\cdot) = 0,\ \partial f_1(k,\cdot)/\partial k = 0\}$ has Lebesgue measure zero.

Saddle point stability of the BGP if it is unique in case (ii). Note that sign det $J =$ sign $(\partial f(k,\cdot)/\partial k)(1-\varphi\bar{k})$ holds. $\eta > \delta$, $\delta - (\rho+\delta)/\sigma \geq 0$ and $\bar{k} \in (\varphi^{-1}, \infty)$ give

$$\lim_{k \searrow \varphi^{-1}} f(k,\cdot) = -\infty \quad \text{and} \quad \lim_{k\to\infty} f(k,\cdot) = \delta - (\rho+\delta)/\sigma \geq 0, \tag{20}$$

$$\lim_{k \searrow \varphi^{-1}} \frac{\partial f(k,\cdot)}{\partial k} = +\infty \quad \text{and} \quad \lim_{k\to\infty} \frac{\partial f(k,\cdot)}{\partial k} = 0 \tag{21}$$

($\searrow$ means that k approaches φ^{-1} from above). (20)-(21) show that $\partial f(k,\cdot)/\partial k > 0$ at the intersection point, and this point is in the range $k \in (\varphi^{-1}, \infty)$. Consequently, det $J < 0$. If $\delta - (\rho+\delta)/\sigma = 0$, $f(k,\cdot)$ must intersect the horizontal axis from below and then converge to zero. Since there may exist an inflection point for $f(k,\cdot)$ for $\eta - \delta > 0$ and $k > \varphi^{-1}$ this possibility is given.

Two BGPs. $\eta - \delta > 0$, $\delta - (\rho+\delta)/\sigma < 0$ and $\bar{k} \in (\varphi^{-1}, \infty)$ hold. For $\eta - \delta > 0$, now $f(k,\cdot)$ has the following properties:

$$\lim_{k \searrow \varphi^{-1}} f(k,\cdot) = -\infty \quad \text{and} \quad \lim_{k\to\infty} f(k,\cdot) = \delta - (\rho+\delta)/\sigma < 0, \tag{22}$$

$$\lim_{k \searrow \varphi^{-1}} \frac{\partial f(k,\cdot)}{\partial k} = +\infty \quad \text{and} \quad \lim_{k\to\infty} \frac{\partial f(k,\cdot)}{\partial k} = 0, \tag{23}$$

Since $\bar{k} \in (\varphi^{-1}, \infty)$, $1 - \varphi\bar{k} < 0$ and $\partial f(k,\cdot)/\partial k > 0$ at the first (lower) $\bar{k}$ and $\partial f(k,\cdot)/\partial k < 0$ at the second (larger) $\bar{k}$. Consequently, det $J < 0$ for the BGP with the lower $\bar{k}$ (higher growth rate), and det $J > 0$ for the BGP with the higher value of $\bar{k}$ (lower growth rate). Thus, the proof is completed.