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# An endogenous growth model with public capital and sustainable government debt

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## Abstract

This paper presents and analyzes an endogenous growth model with public capital and public debt. It is assumed that the ratio of the primary surplus to gross domestic income is a positive linear function of the debt income ratio which assures that public debt is sustainable. The paper then derives necessary conditions for the existence of a sustainable balanced growth path for the analytical model. Further, simulations are undertaken in order to gain insights into stability properties of the model and in order to analyze growth effects of deficit financed increases in public investment. The latter is done for the model on the sustainable balanced growth path as well as for the model along the transition path.

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# 1  Introduction

One strand in endogenous growth theory postulates that sustained per-capita growth results from ongoing investment in public capital which raises the incentive of private investors to build up a private capital stock. Productive public capital has a long tradition in the economics literature. Arrow and Kurz [1] were among the first to present a formal model with that type of capital. However, their approach did not allow for sustained per-capita growth in the long-run. Futagami et al. [10] then presented an endogenous growth model with productive public capital which generates sustained per-capita growth in the long-run. Their model basically is a more general version of the simple approach presented by Barro [4]. The difference between these two models is that Futagami et al. assume that public investment does not affect aggregate production possibilities directly, as does Barro, but only indirectly by building up a stock of public capital which stimulates economic production.

One consequence of the model presented by Futagami et al. is that their model gives rise to transition dynamics, which does not hold for the Barro model. Both models, however, have in common that the budget of the government is balanced at any moment in time, as frequently assumed in this class of models. Further, as a consequence of productive public capital there exists a growth maximizing income tax rate with an inverted U-relationship between the balanced growth rate and the size of the income tax rate. In Greiner and Hanusch [13] the model by Futagami et al. is extended and allows for both productive and non-productive public spending and it is demonstrated that growth and welfare maximization may be different even if one confines the investigation to the balanced growth path.

In the approaches mentioned above the public capital stock is a purely public good which is non-rivalrous and non-excludable. Glomm and Ravikumar [12] in their review of the literature present a model where this issue is addressed among others. Further, they explicitly distinguish between government expenditures which enter as inputs in the

production function for output and expenditures which raise the productivity of investment technologies. Baier and Glomm [3] extend the approach by allowing for an elasticity of substitution between public and private capital which is not necessarily equal to one. These authors demonstrate that the elasticity of substitution affects the growth maximizing ratio between private and public capital and, as a consequence, the growth maximizing tax rates on capital and labour.

As concerns the empirical relevance of public capital for the productivity of economies the results are not unambiguous. A frequently cited study is the paper by Aschauer [2], for example, who reports strong effects of public capital. Further, he states that public capital is dramatically more important than public investment as a flow variable. However, there are also studies which reach different conclusions. This is not too surprising because it is to be expected that the time period under consideration as well as the countries which are considered are important as to the results obtained. For a survey of the empirical studies dealing with public spending, public capital and the economic performance of countries see Sturmet al. [20] and Pfähler et al. [19].

All what the theoretical models have in common is that they assume a balanced government budget. An exception to this assumption is provided by the model presented by Turnovsky [22], chap. 13, who allowed for public debt in his analysis. He demonstrates that an increase in public investment financed by higher public debt unambiguously raises the balanced growth rate (p. 418). The reason for that outcome is that public capital stimulates investment and public debt does not affect the allocation of resources in the long-run and, therefore, does not have negative growth effects.

On the other hand, public debt and the question of whether public debt is sustainable plays an important role in market economies. So, the latter question has been the subject of a great many empirical studies, in particular as concerns the U.S. (see e.g. Hamilton and Flavin [15], Kremers [17], Wilcox [24], or Trehan and Walsh [21]). However, no unambiguous answer could be obtained and Bohn [7], [8] criticized these tests because they

make assumptions about future states of nature that are difficult to estimate from a single set of observed time series data. Therefore, he proposes a different test which analyzes whether the ratio of the primary surplus to gross domestic product is a positive linear function of the ratio of public debt to gross domestic product which guarantees sustainability of public debt. The reasoning behind this argument is that if a government raises the primary surplus as public debt increases it takes corrective actions which stabilize the debt ratio. This implies that the debt ratio displays mean-reversion and thus the ratio remains bounded implying that public debt is sustainable.

The empirical analysis for the U.S. indeed confirms that a higher debt ratio leads to higher primary surpluses (cf. Bohn [8]). The same also holds for countries in the EURO area (see Greiner et al. [14]). Thus, the intertemporal budget constraint, although it should be fulfilled only in infinity, has immediate repercussions for the period budget constraint since the government reduces public spending or/and raises tax revenues as public debt rises.

In this paper, we present a theoretical model where we combine the two topics mentioned above. That is we present an endogenous growth with public investment following the approach by Futagami et al. [10] and we integrate public debt. Further, we assume that the primary surplus of the government is a positive linear function of public debt which guarantees that the intertemporal budget constraint of the government holds. Given this assumption the paper then analyzes the structure of model where we pay particular attention to the dynamic behaviour.

The rest of the paper is organized as follows. In the next section we demonstrate that sustainability of public debt is given if the primary surplus is a positive linear function of public debt. In section 3 we present the endogenous growth model with public capital and government debt. Section 4 studies the implications of the model and analyzes growth effects of deficit financed increases in public investment both for the model on the sustainable growth path and taking into account transition dynamics. Section 5, finally,

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concludes the paper.

# 2 The primary surplus and sustainability of public debt

The accounting identity describing the accumulation of public debt in continuous time is given by:

$$\dot{B}(t) = B(t)r(t) - S(t), \tag{1}$$

where $B(t)$ stands for real public debt,[1] $r(t)$ is the real interest rate, and $S(t)$ is real government surplus exclusive of interest payments.

Solving equation (1) we get for the level of public debt at time $t$

$$B(t) = e^{\int_0^t r(\tau)d\tau} \left( B(0) - \int_0^t e^{-\int_0^\tau r(\mu)d\mu} S(\tau)d\tau \right), \tag{2}$$

with $B(0)$ public debt at time $t$. Multiplying both sides of (2) with $e^{-\int_0^t r(\tau)d\tau}$, to get the present value of government debt at time $t$, yields

$$e^{-\int_0^t r(\tau)d\tau} B(t) + \int_0^t e^{-\int_0^\tau r(\mu)d\mu} S(\tau)d\tau = B(0). \tag{3}$$

If the first term in (3) goes to zero in the limit the current value of public debt equals the sum of discounted future non-interest surpluses. Then, we have

$$B(0) = \int_0^t e^{-\int_0^\tau r(\mu)d\mu} S(\tau)d\tau. \tag{4}$$

Equation (4) is the present-value borrowing constraint and we call a path of public debt which satisfies this constraint a sustainable debt. It states that public debt at time zero must equal the future present-value surpluses. Equivalent to requiring that (4) must be fulfilled is that the following condition holds:

$$\lim_{t \to \infty} e^{-\int_0^t r(\tau)d\tau} B(t) = 0. \tag{5}$$

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[1]Strictly speaking, $B(t)$ should be real public net debt.

That equation is usually referred to as the no-Ponzi game condition (see e.g. Blanchard and Fischer (1989), ch. 2).

Now, assume that the ratio of the primary surplus to gross domestic income ratio is a positive linear function of the debt to gross domestic income ratio and of a constant. The primary surplus ratio, then, can be written as

$$\frac{T(t) - I_p(t)}{Y(t)} = \phi + \beta \frac{B(t)}{Y(t)}, \tag{6}$$

where $T(t)$ denotes the tax revenue at time $t$, $I_p(t)$ is public spending at $t$, $Y(t)$ gross domestic income at $t$ and $\phi, \beta \in \mathbb{R}$ are constants. All variables are real variables. It should be noted that $\beta$ determines how strong the primary surplus reacts to changes in public debt and, therefore, can be considered as a feedback parameter of public debt. $\phi$ determines whether the level of the primary surplus rises or falls with an increase in gross domestic income.

Using that equation the differential equation describing the evolution of public debt can be written as

$$\dot{B}(t) = r(t)\, B(t) - T(t) + I_p(t) = (r(t) - \beta)\, B(t) - \phi\, Y(t). \tag{7}$$

Solving this differential equation and multiplying both sides with $e^{-\int_0^t r(\tau)d\tau}$ to get the present value of public debt yields

$$e^{-\int_0^t r(\tau)d\tau} B(t) = e^{-\beta t} \left( B(0) - \phi\, Y(0) \int_0^t e^{\beta\tau - \int_0^\tau (r(\mu) - \gamma_y(\mu))d\mu} d\tau \right), \tag{8}$$

with $B(0)$ public debt at time $t = 0$ and $\gamma_y$ the growth rate of gross domestic income.

First, we state that for $r < \gamma_y$ the intertemporal budget constraint is irrelevant because in this case the economy is dynamically inefficient implying that the government can play a Ponzi game. Therefore, we only consider the case $r > \gamma_y$.

Writing equation (8) as

$$e^{-\int_0^t r(\tau)d\tau} B(t) = e^{-\beta t} B(0) - \phi\, Y(0) \frac{\int_0^t e^{\beta\tau}\, e^{-\int_0^\tau (r(\mu) - \gamma_y(\mu))d\mu} d\tau}{e^{\beta t}}. \tag{9}$$

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shows that $\beta > 0$ is a necessary condition for $\lim_{t\to\infty} e^{-\int_0^t r(\tau)d\tau} B(t)$, i.e. for the present value of public debt to converge to zero for $t \to \infty$.

If the numerator in the second expression in (9) remains finite, implying that $\int_0^\tau (r(\mu) - \gamma_y(\mu))d\mu$ converges to infinity, the second term converges to zero. If the numerator in the second expression in (9) becomes infinite, l'Hôpital gives the limit as $e^{-\int_0^t (r(\mu)-\gamma_y(\mu))d\mu}/\beta$. This shows that $\beta > 0$ and $\lim_{t\to\infty} \int_0^t (r(\mu) - \gamma_y(\mu))d\mu = \infty$ are sufficient for sustainability of public debt.

These considerations demonstrate that the intertemporal budget constraint of the government is fulfilled if the ratio of the primary surplus to gross domestic income is a positive linear function of the debt ratio, which can also be observed for economies in the real world. Therefore, we posit that the government sets the primary surplus according to (6) implying that public debt is sustainable. In the next section, we present our endogenous growth model with public capital and with that assumption.

# 3    The structure of the growth model

Our economy consists of three sectors: A household sector which receives labour income and income from its saving, a productive sector and the government. First, we describe the household and the productive sector.

## 3.1    The household and the productive sector

The household sector is represented by one household which maximizes the discounted stream of utility resulting from per-capita consumption, $C$,[2] over an infinite time horizon subject to its budget constraint. The utility function is assumed to be logarithmic, $U(C) = \ln C$, and the household has one unit of labour, $L$, which it supplies inelastically. The

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[2] From now on we omit the time argument $t$ if no ambiguity arises.

maximization problem, then, can be written as

$$\max_{C} \int_{0}^{\infty} e^{-\rho t} \ln C \, dt, \tag{10}$$

subject to

$$(1 - \tau)(w + rW + \pi) = \dot{W} + C. \tag{11}$$

$\rho$ is the subjective discount rate, $w$ is the wage rate, $r$ is the interest rate and $\pi$ gives possible profits from the productive sector which the household takes as given in solving its optimization problem. $W \equiv B + K$ denotes assets which are equal to public debt, $B$, and private capital, $K$. All variables give per-capita quantities. $\tau \in (0,1)$ is the income tax rate. The dot gives the derivative with respect to time and we neglect depreciation of private capital.

To solve this problem we formulate the present-value Hamiltonian which is written as

$$\mathcal{H} = \ln C + \lambda((1 - \tau)(w + rW + \pi) - C) \tag{12}$$

Necessary optimality conditions are given by

$$C^{-1} = \lambda \tag{13}$$

$$\dot{\lambda} = \rho\lambda - \lambda(1 - \tau)r \tag{14}$$

If the transversality condition $\lim_{t \to \infty} e^{-\rho t} W/C = 0$ holds which is fulfilled for a time path on which assets grow at the same rate as consumption the necessary conditions are also sufficient.

The productive sector is represented by one firm which behaves competitively and which maximizes static profits. The production function of the firm is given

$$Q = K^{1-\alpha} G^{\alpha} L^{\xi}, \tag{15}$$

with $\alpha + \xi \leq 1$. $(1 - \alpha)$ is the private capital share, $\alpha$ gives the public capital share and $\xi$ is the labour share. $G$ denotes public capital which is assumed to be a purely public

good. Using that labour is normalized to one profit maximization yields

$$w = \xi K^{1-\alpha} G^{\alpha} \tag{16}$$

$$r = (1-\alpha) K^{-\alpha} G^{\alpha} \tag{17}$$

Resorting to (13), (14) and (16), (17), which must hold in equilibrium, the growth rate of consumption is derived as

$$\frac{\dot{C}}{C} = -\rho + (1-\tau)(1-\alpha) K^{-\alpha} G^{\alpha}. \tag{18}$$

### 3.1.1 The government

The government in our economy receives tax revenues from income taxation and has revenues from issuing government bonds it then uses for public investment. Further, the primary surplus is a positive linear function of public debt which guarantees that public debt is sustainable as shown in the previous section.

Using (6) and defining $i_p \equiv (1 - \phi/\tau)$, the budget constraint of the government can be written as

$$\dot{B} + T = rB + I_p \leftrightarrow \dot{B} = (r - \beta)B + T(i_p - 1), \tag{19}$$

with $\beta > 0$ and $I_p = i_p T - \beta B$ public investment which amounts to total public spending. It should be noted that we have assumed that $i_p$ denotes that fraction of the tax revenue the government uses for gross public investment. $i_p < 1$ implies that a certain part of the tax revenue is used for the debt service and $i_p > 1$ implies that no part of the tax revenue is used for the debt service and public investment may exceed the tax revenue.

Neglecting depreciation, public capital evolves according to

$$\dot{G} = I_p = i_p T - \beta B. \tag{20}$$

## 3.2 Equilibrium conditions and the sustainable balanced growth path

An equilibrium allocation is defined as an allocation such that the firm maximizes profits implying that factor prices equal their marginal products (equations (16) and (17)), the household solves (10) subject to (11) and the budget constraint of the government (19) is fulfilled.

The economy-wide resource constraint is obtained by combining equations (11) and (19) as

$$\frac{\dot{K}}{K} = -\frac{C}{K} + \frac{K^{1-\alpha}G^{\alpha}}{K} - \left( i_p \frac{T}{K} - \beta \frac{B}{K} \right) . \tag{21}$$

Thus, the economy is completely described by equations (18), (19), (20) and (21) plus the limiting transversality condition of the household.

A sustainable balanced growth path (SBGP) is defined as a path on which all endogenous variables grow at the same rate, i.e. $\dot{K}/K = \dot{G}/G = \dot{B}/B = \dot{C}/C$ holds, and the intertemporal budget constraint of the government is fulfilled, that is equation (5) must hold. Note that the SBGP is dynamically efficient[3] and the transversality condition of the household is fulfilled. Since we have posited that the government sets the primary surplus according to (6) with $\beta > 0$ any path which satisfies $\dot{K}/K = \dot{G}/G = \dot{B}/B = \dot{C}/C$ is associated with a sustainable public debt. To make this clear we speak of a sustainable balanced growth path.

To analyze our economy around a SBGP we define the new variables $x \equiv G/K$, $b \equiv B/K$ and $c \equiv C/K$. Differentiating these variables with respect to time yields a

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[3]The difference between the interest rate and the growth on the SBGP is strictly positive and constant implying $\lim_{t \to \infty} \int_0^t (r(\mu) - \gamma_y(\mu)) d\mu = \infty$.

three dimensional system of differential equations given by

$$\dot{x} = x\left(c - \beta b(1 + x^{-1}) - x^{\alpha} + (1 + (1 - \alpha)b)i_p \tau x^{\alpha}(1 + x^{-1})\right), \tag{22}$$

$$\dot{b} = b\left(c - \beta(1 + b) + (1 - \alpha)x^{\alpha} + (i_p - 1)\tau x^{\alpha}((1 - \alpha) + b^{-1}) - x^{\alpha} + \right.$$

$$\left. i_p \tau \, x^{\alpha}(1 + (1 - \alpha)b)\right), \tag{23}$$

$$\dot{c} = c\left(c - \rho + (1 - \tau)(1 - \alpha)x^{\alpha} - x^{\alpha} - \beta b + i_p \tau x^{\alpha}(1 + (1 - \alpha)b)\right). \tag{24}$$

A solution of $\dot{x} = \dot{b} = \dot{c} = 0$ with respect to $x, b, c$ gives a SBGP for our model and the corresponding ratios $x^{\star}, b^{\star}, c^{\star}$ on the SBGP.[4] In the next section we first analyze the structure of our model and, then, investigate how deficit financed increases in public investment affect the balanced growth rate and the growth rate on the transition path.

# 4 Implications of the model

To get insight into our model we first solve (24) with respect to $c$ and insert that value in (23) giving

$$\frac{\dot{b}}{b} = (\rho - \beta) + (i_p - 1)\tau x^{\alpha}((1 - \alpha) + b^{-1}) + (1 - \alpha)\tau x^{\alpha}. \tag{25}$$

From equation (25) we can derive a first result.

Assume that the government is a debtor, that is $b > 0$ holds. Then, for $\beta \leq \rho$ the right hand side in that equation can become zero, which is necessary for a SBGP to exist, only if $i_p < 1$. $i_p < 1$ is given if the level of the primary surplus rises with an increase in gross domestic income, i.e. $\phi > 0$ must holds From an economic point of view this implies that a certain part of the tax revenue must be used for the debt service if the economy is to grow over time at a constant rate if $\beta$ is relatively small, that is if the primary surplus does not increase sufficiently as public debt rises. Relatively small means that the parameter $\beta$ is lower than the rate of time preference $\rho$. But this result only holds if the government

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[4]The $\star$ denotes SBGP values and we exclude the economically meaningless SBGP $x^{\star} = b^{\star} = c^{\star} = 0$.

is a debtor, i.e. for $b > 0$. This is obvious because there is no need for the government to reduce a possibly existing primary deficit if the government is a net lender.

For $\beta > \rho$, a SBGP can exist for $i_p > 1$ and a positive public debt. $i_p > 1$ implies that the level of the primary surplus negatively depends on gross domestic income, i.e. $\phi < 0$ holds. In this case, the reaction of the government to a higher debt ratio, modelled by the parameter $\beta$, is sufficiently strong so that an increase in may go along with a reduction in the primary surplus.

These considerations have given some first insights into our model. In the next subsection we will further pursue the question of whether a SBGP exists and whether it is stable. In addition, we will analyze growth effects of deficit financed increases in public investment for the model on the SBGP. With deficit financed increase in public investment we mean an increase in public investment, modelled by a rise in $i_p$, which does not go along with a higher income tax rate.

## 4.1   The economy on the SBGP

To analyze our model further, we resort to simulations. We do so because the analytical model turns out to become too complex to derive further results. As a benchmark for our simulations we set the income tax rate to ten percent, i.e. $\tau = 0.1$, the elasticity of production with respect to public capital is set to 25 percent, i.e. $\alpha = 0.25$.[5] The rate of time preference is set to 30 percent, $\rho = 0.3$. Interpreting one time period as 3 (5, 10) years then gives an annual rate of time preference of 10 (6, 3) percent.

In table 1 we report results of our simulations for values of $\beta$ which are smaller than the rate of time preference $\rho$. $\gamma$ denotes the balanced growth rate and unstable means that at least two eigenvalues are positive or have positive real parts.

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[5]For a survey of empirical studies giving estimates for that parameter see Pfähler [19] or Sturm [20].

Table 1

| $i_p$ | $\beta = 0.15$ | | | | $\beta = 0.25$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $b^\star$ | $x^\star$ | $\gamma$ | Stability | $b^\star$ | $x^\star$ | $\gamma$ | Stability |
| $i_p = 1.15$ | -0.05 | 0.4 | 0.238 | unstable | -0.1 | 0.45 | 0.252 | unstable |
| $i_p = 1.05$ | -0.02 | 0.37 | 0.226 | unstable | -0.04 | 0.39 | 0.232 | unstable |
| $i_p = 0.9$ | 0.04 | 0.31 | 0.204 | unstable | 0.07 | 0.27 | 0.186 | unstable |
| $i_p = 0.75$ | 0.09 | 0.24 | 0.174 | unstable | 0.16 | 0.05 | 0.022 | unstable |
| | | | | | 0.15 | 0.04 | 0 | unstable |
| $i_p = 0.45$ | 0.15 | 0.04 | 0 | unstable | no SBGP for $i_p \leq 0.74$ | | | |
| | 0.11 | 0.01 | -0.088 | unstable | | | | |
| | no SBGP for $i_p \leq 0.44$ | | | | | | | |

Table 1 confirms the result derived for the analytical model that for $\beta < \rho$ a certain part of the tax revenue must be used for the debt service, i.e. $i_p < 1$ must hold, to get sustained growth if public debt is positive. For $i_p > 1$ sustained growth goes along with a negative government debt, that is the government must be a creditor.

Further, one realizes that the smaller $i_p$, i.e. the smaller that part of the tax revenue used for public investment, the smaller is the balanced growth rate $\gamma$ in case where the SBGP is unique. This implies that raising public investment increases the balanced growth rate. If that part of the tax revenue which is used for public investment, $i_p$, falls below a certain critical value the model does not yield sustained growth at all. This critical value is the larger the larger the parameter $\beta$. From an economic point of view this is obvious because a high $\beta$ implies that a given level of public debt goes along with a low level of public investment since a large fraction of public revenues is used for the debt service.

As to stability, the SBGP is unstable in all cases. The eigenvalues are real with two being positive and one being negative. This means that there exists a one dimensional stable manifold. If one takes $x(0)$ and $b(0)$ as given this implies that the set of initial

conditions $\{x(0), b(0), c(0)\}$ lying on the stable manifold has Lebesgue measure zero. In this case the economy can converge to the SBGP in the long-run only if the government levies a lump-sum tax at $t = 0$ which is used to control $B(0)$ implying that $B(0)$, and thus $b(0)$, can be set. $B(0)$ and $C(0)$, then, must be chosen such that $b(0)$ and $c(0)$ lie on the stable manifold and these values are uniquely determined. In addition, for small values of $i_p$ two SBGPs exist where one goes along with a zero or negative growth rate, respectively.[6]

To gain further insight into our model we next set $\beta > \rho$. The results of the simulations are shown in table 2.

Table 2

| $i_p$ | $\beta = 0.35$ | | | | $\beta = 0.4$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $b^\star$ | $x^\star$ | $\gamma$ | Stability | $b^\star$ | $x^\star$ | $\gamma$ | Stability |
| $i_p = 1.15$ | -0.47 | 0.72 | 0.322 | unstable | 0.16 | 0.1 | 0.077 | stable |
| $i_p = 1.05$ | -0.24 | 0.56 | 0.283 | unstable | 0.08 | 0.26 | 0.181 | stable |
| | 0.15 | 0.04 | 0 | unstable | | | | |
| $i_p = 1.02$ | -0.13 | 0.47 | 0.258 | unstable | 0.04 | 0.31 | 0.205 | stable |
| | 0.16 | 0.1 | 0.08 | stable | | | | |
| $i_p = 0.9$ | no SBGP for $i_p < 1$ | | | | -0.2 | 0.53 | 0.275 | stable |
| $i_p = 0$ | no SBGP for $i_p < 1$ | | | | -1.01 | 1.06 | 0.384 | stable |

To interpret table 2 we first consider the case $\beta = 0.35$.

One can see that, as in table 1, the balanced growth rate is the larger the larger the fraction of the tax revenue which is used for public investment, $i_p$, for the unstable SBGP. Further, there exists a critical value for $i_p$ below which sustained growth is not feasible. This critical value is larger than in table 1 because of the higher $\beta$. In addition, for about

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[6]A negative growth rate of public capital does not make sense in our model. It would imply that public capital is sold to the private sector.

$i_p \in (1, 1.07)$ there exist two SBGPs. The unstable SBGP has one negative and two positive real eigenvalues, the stable SBGP has one positive real eigenvalue and a pair of complex conjugate eigenvalues with negative real parts. The balanced growth rate of the stable SBGP negatively depends on $i_p$ in contrast to the growth rate associated with the unstable SBGP which depends positively on $i_p$.

As to stability, the stable SBGP looses stability with a rising value of $i_p$. For about $i_p \in (1, 1.028)$ the SBGP is stable and for $i_p > 1.028$ the stable SBGP becomes unstable and this SBGP disappears for $i_p > 1.07$ leaving only the unstable SBGP. For $i_p = i_p^{crit} = 1.028651$ the stable SBGP undergoes a Hopf bifurcation and leads to unstable limit cycles.[7] For a slightly different value of $\rho$, namely for $\rho = 0.32$,[8] a supercritical Hopf bifurcation can be observed for $i_p = i_p^{crit} = 0.971824$ which leads to stable limit cycles. In this case, there exists an interval of $i_p$ with strictly positive measure for which the economy does not converge to the SBGP but converges to persistent cycles. The limit cycles occur for values of $i_p$ larger $i_p^{crit}$. From an economic point of view this means that the economy is characterized by sustained fluctuations around the SBGP. Figure 1 shows the limit cycle in the $(x-b-c)$ phase space where the orientation is counter clockwise as indicated by the arrows.

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[7] For those computations we used the software LOCBIF, see Khibnik et al. [18], and MATCONT, see Dhooge et al. [9].

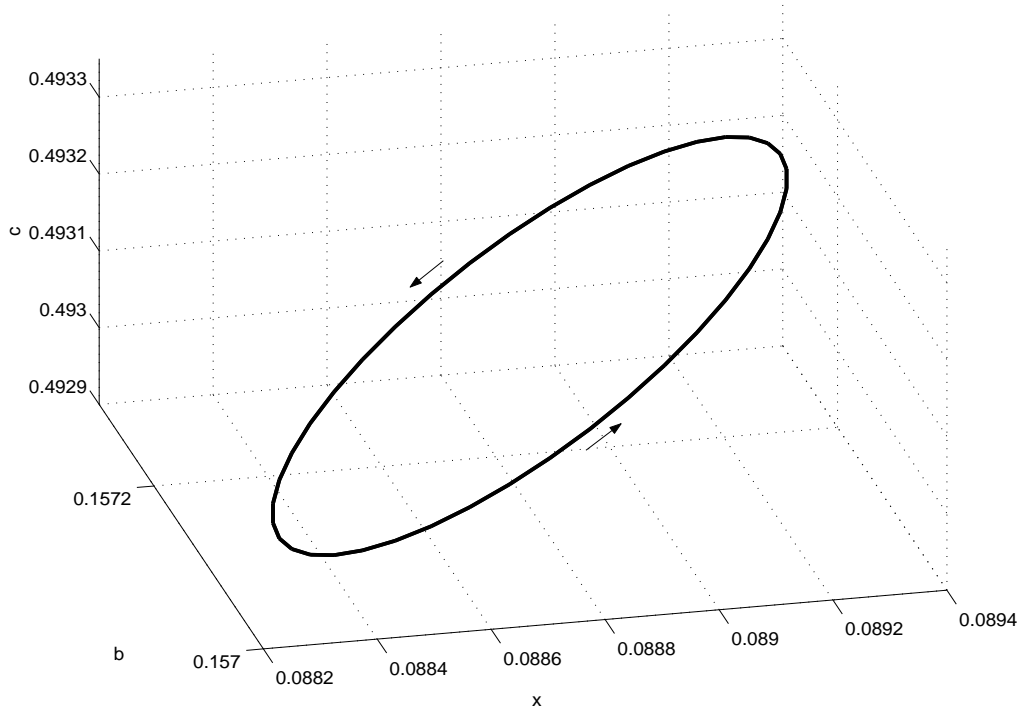[8] With $\rho = 0.32$ there exist two SBGPs for about $i_p \in (0.96, 1)$.

Figure 1: Limit cycle in the $(x - b - c)$ phase space.

To understand the emergence of limit cycles from an economic point of view, we assume that the economy originally is on the SBGP. The government, then, raises $i_p$ such that this parameter falls in that interval of $i_p$ which generates stable cycles. As a consequence of the increase in $i_p$, public investment rises leading to an increase in the growth rate of public capital and in the ratio $G/K = x$. The latter increase raises the marginal product of private capital and leads to a higher growth rate of private consumption and of the ratio $C/K = c$. As a result of the increase in $i_p$, however, public debt also rises implying that the resources for the debt service increase leading to a rise in the ratio $B/K = b$. From figure 1 it can be seen that $b$ lags behind $x$ which makes sense from an economic point of view. The increase in resources required for the debt service, finally, leads to a decrease in the growth rate of public capital and in the ratio $G/K = x$. The latter effect reduces the marginal product of private capital and leads to a smaller growth rate

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of private consumption and to a decline in the ratio $C/K = c$. When the public debt ratio has fallen enough public investment rises again which spurs economic growth.

In this way, a cyclical evolution is generated. It should be noted that for lower values of $\beta$ these fluctuations cannot be observed because then $\beta$ is not sufficiently high to stabilize the economy. For larger values of $\beta$ cycles are excluded, too, because high values of $\beta$ tend to stabilize the economy in a way that it always converges to the SBGP as we will see next.

Next, we consider the case $\beta = 0.4$.

Table 2 shows that, in this case, there exists a unique SBGP which is stable. The high value of $\beta$ guarantees that the primary surplus of the government reacts sufficiently strong to higher public debt which stabilizes the economy. The eigenvalues are real with two being negative and one being positive implying that there exists a two-dimensional stable manifold and a unique $c(0)$ so that the economy converges to the SBGP in the long-run. Now, however, the balanced growth rate negatively depends on $i_p$. In this case, a deficit financed increase in public investment is offset by the higher public debt, which requires more resources for the debt service, so that the economy finally invests less in public capital. Sustained growth is also given for small values of $i_p$ and even for $i_p = 0$. But again, the government must be a creditor in this case. For $i_p = 0$ public investment is completely financed by public wealth which increases over time due to interest payments and due to the tax revenue. The same outcome is observed for $\beta = 0.5$.

Before we go on with our analysis we briefly summarize our results obtained from the simulations up to now. We saw that the higher $\beta$, i.e. the stronger the primary surplus and, thus public spending, react to increases in public debt, the sooner the model is stable. In this case, a deficit financed increase in public investment reduces the balanced growth rate because of the strong feedback effects associated with public debt. It could also be shown that the model may be very sensitive with respect to $\beta$, the feedback parameter of public debt, and with respect to $i_p$, giving that part of the tax revenue used for public

investment. So, variations in these two parameters may lead to stable limit cycles and multiple SBGPs.

Further, for small values of $\beta$ there exists a critical value of $i_p$ below which sustained growth is not possible. This critical value is the large the larger is $\beta$ which makes sense from an economic point of view. Finally, it could be realized that for $\beta > \rho$ that part of the tax revenue used for public investment, $i_p$, must be smaller than one to achieve sustained growth unless the government is a creditor. A fact already shown for the analytical model.

**Robustness of the results**

Before we study the model along the transition path we investigate whether changing the numerical parameter values affects the qualitative outcome. To do so we first set $\alpha = 0.15$ and leave $\tau$ and $\rho$ unchanged. Then, we set $\alpha = 0.15$ and $\tau = 0.3$ and leave $\rho$ unchanged. Finally, we change all three parameters and set $\alpha = 0.15$, $\tau = 0.3$ and $\rho = 0.03$.

Performing the calculations which led to tables 1 and 2 with the other parameter values shows that the qualitative results do not change.[9] That is, for small values of $\beta$ the model is unstable and there exists a critical value for $i_p$ below which sustained growth is not feasible. The balanced growth rate associated with this SBGP positively depends on $i_p$, the fraction of the tax revenue used for public investment.

For higher values of $\beta$ two SBGPs exist with one being unstable and the other being stable. Further, the stable SBGP looses stability as $i_p$ is increased and becomes unstable before it vanishes when $i_p$ is further increased. When $\beta$ is increased further there exists a unique SBGP which is stable. The balanced growth rate associated with this SBGP negatively depends on $i_p$.

Although the qualitative outcome does not change the values of $\beta$ and $i_p$ which generate the respective results and the numerical values of the endogenous variables are different. Of course, this was to be expected. However, since we are interested in qualitative features

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[9]Tables reporting the exact outcome are available on request.

of our model this is of less importance.

## 4.2   Fiscal policy on the transition path

In this subsection we study the behaviour of our model after a once-and-for all deficit financed increase in public investment taking into account transition dynamics. We assume that the economy is originally on the SBGP when this fiscal policy is performed at time $t = 0$. Further, we consider the case where our model is characterized by a saddle point with two negative real eigenvalues.

To analyze the effects of a deficit financed increase in public investment we study the solution of the linearized system of (22)-(24) which is given by

$$
\begin{aligned}
x(t) &= x^\star + C_1\, v_{11}\, e^{\mu_1 t} + C_2\, v_{21}\, e^{\mu_2 t}, & (26) \\
b(t) &= b^\star + C_1\, v_{12}\, e^{\mu_1 t} + C_2\, v_{22}\, e^{\mu_2 t}, & (27) \\
c(t) &= c^\star + C_1\, v_{13}\, e^{\mu_1 t} + C_2\, v_{23}\, e^{\mu_2 t}, & (28)
\end{aligned}
$$

with $v_{ij}$ the $j-$th element of the eigenvector belonging to the negative real eigenvalue $\mu_i$, $i = 1, 2$. $C_i$, $i = 1, 2$, are constants determined by the initial conditions $x_0$ and $b_0$. Setting $t = 0$ gives $C_i$, $i = 1, 2$, as a function of $x_0$ and $b_0$. Inserting these $C_i$, $i = 1, 2$, in (28) gives the unique $c(0)$ on the stable manifold leading to the SBGP in the long-run. Given $x(t)$, $b(t)$ and $c(t)$ one can compute the growth rates of $C$, $B$, $G$ and $K$ according to (18)-(21).

To analyze growth effects of a deficit financed increase in public investment we take the numerical example from the last subsection with $\beta = 0.4$ and raise $i_p$ from $i_p = 1.05$ to $i_p = 1.055$ which reduces the long-run balanced growth rate as can be seen from the right part of table 2. The stable manifold of the linearized system and the adjustment to the new SBGP are shown in Figure 2
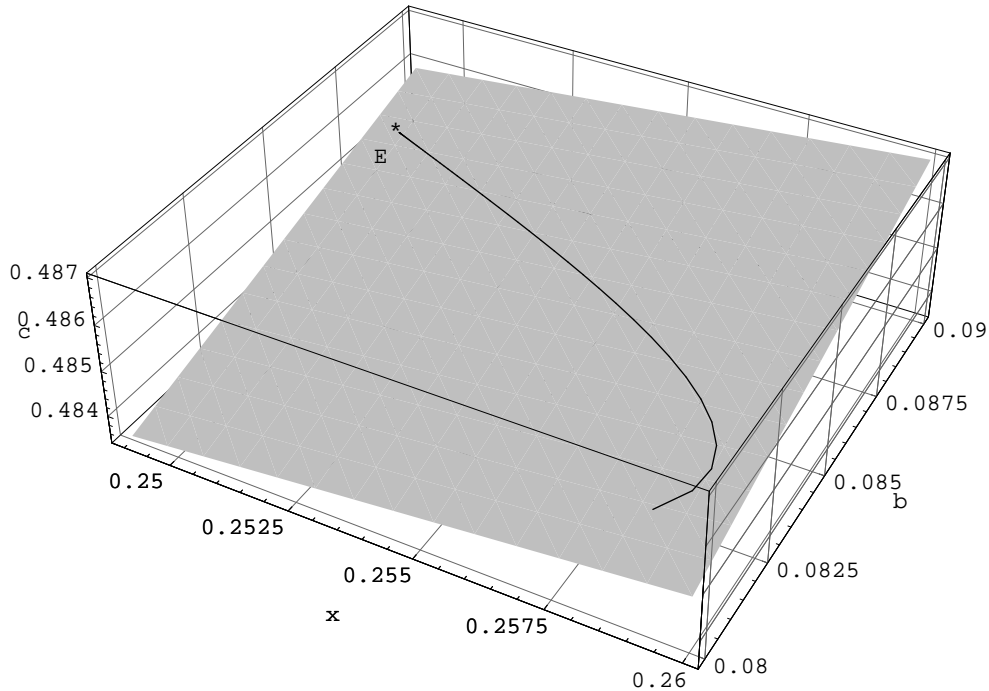
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Figure 2: The stable manifold and the transition path to the new SBGP denoted by $E$.

At time $t = 0$, the ratio $c$ jumps from the old SBGP value 0.4849 to $c(0) = 0.4848$ onto the stable manifold and then rises. Over time the path approaches the new SBGP given by $E = (x^\star, b^\star, c^\star) = (0.2497, 0.0882, 0.4855)$. One can see that $x$ first rises and then declines while $b$ monotonically rises. $c$ monotonically rises for $t > 0$, i.e. after the initial downward adjustment at $t = 0$.

The reaction of the growth rates of private capital (solid line) and of public capital (dotted line) to the increase in $i_p$ are shown in figure 3. For $t < 0$, the solid line gives the balanced growth rate before the increase in $i_p$.
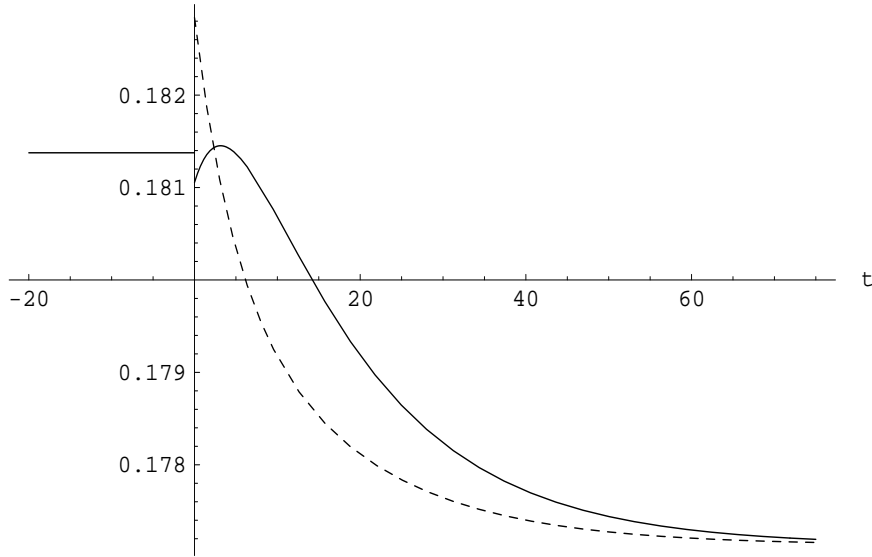
Figure 3: $\dot{G}/G$ (dotted line) and $\dot{K}/K$ (solid line) on the transition path.

First, we state that $x$ and $b$ do not react to parameter changes at time $t = 0$ since $K$, $G$ and $B$ are predetermined variables which react only gradually while $C$ immediately reacts so that $c$ jumps to the stable manifold. From figure 3 one realizes that an increase in $i_p$ raises the growth rate of public capital, $\dot{G}/G$, which jumps upward at $t = 0$. (20) shows that a rise in $i_p$ must raise $\dot{G}/G$ at $t = 0$ since $B$ as well as $K$ and $G$ and, thus the tax revenue $T$, are fixed at $t = 0$. From an economic point of view this reaction is obvious since raising the part of the tax revenue used for investment will cause an immediate increase in public investment. Over time, however, the initial increase in public investment caused by the higher $i_p$ is offset by the increase in public debt and, consequently, the growth rate of public capital declines and approaches the new SBGP. So, figure 3 demonstrates that both growth rates overshoot the long-run balanced growth rate, i.e. they first increase before they decline.

As concerns the growth rate of private capital there are two counteracting effects. On the one hand, the increase in $i_p$ reduces the consumption share $c$ at $t = 0$, which tends

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to raise the growth rate of private capital. On the other hand, the increase in the public deficit, i.e. the rise in $\dot{B}$, implies that there is a crowding-out of private saving at $t = 0$ which can be seen from (11). This effect tends to lower $\dot{K}/K$. Figure 3 shows that the crowding-out effect dominates implying that the growth rate of private capital declines at $t = 0$. Over time, however, the growth rate of private capital then first rises before it declines and approaches the new SBGP. The temporary rise in $\dot{K}/K$ results from the increase in the public debt ratio, $b$, and from the temporary increase in production relative to capital, $x^\alpha$, which imply a positive income effect for the private household. But for $t$ sufficiently large $\dot{K}/K$ falls because $x$ declines over time.

It should be noticed that the rise in $i_p$, which does not have distortions per se, affects the growth rates of public and of private capital at $t = 0$ through the income effect. So, the income effect, which is of no relevance as concerns the long-run balanced growth rate, very well affects the growth rates of economic variables along the transition path.

Figure 4 shows the effects of the increase in $i_p$ on the growth rates of public debt (dotted line) and of private consumption (solid line) where the solid line for $t < 0$ again gives the balanced growth rate before the increase in $i_p$.
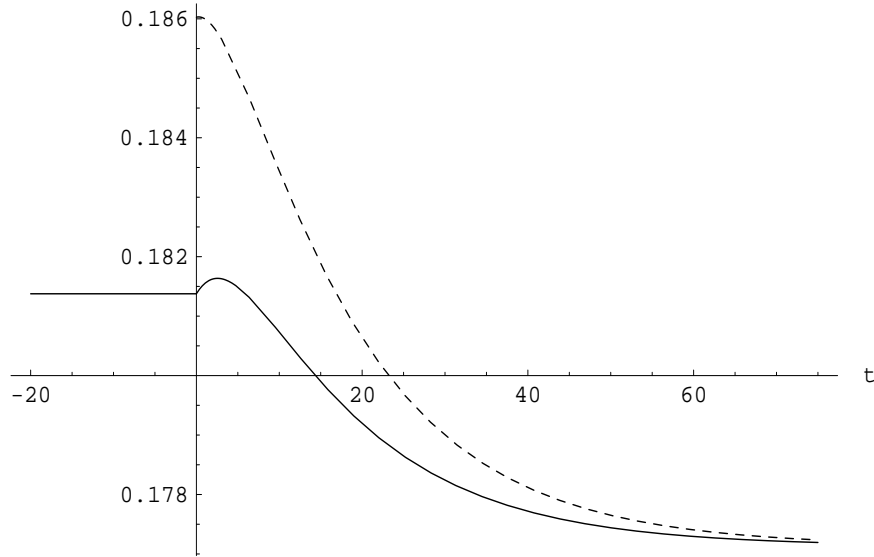
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Figure 4: $\dot{B}/B$ (dotted line) and $\dot{C}/C$ (solid line) on the transition path.

Figure 4 shows that the growth rate of public debt increases at $t = 0$, due to the deficit financed rise of public investment, and then declines and approaches its new SBGP value. From (18) we see that the growth rate of consumption does not react to the increase in $i_p$ at $t = 0$. Over time, $\dot{C}/C$ first rises before it declines. The reason is that, as a consequence of the increase in public investment, the ratio $G/K = x$ first rises and then declines implying the same effect for $\dot{C}/C$. Thus, as in the case of $\dot{G}/G$ and $\dot{K}/K$, the growth rates of public debt and of private consumption first rise and then decline and approach the SBGP value implying an overshooting of the long-run balanced growth rate.

These considerations show that the income effect associated with an increase in public investment affects the growth rates of economic variables on the transition path. This income effect generates an overshooting of the growth rates over the long-run balanced growth rate.

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# 5  Conclusion

This paper has presented an endogenous growth model with public capital and government debt where the government raises the primary surplus as a result of higher public debt. In dynamic efficient economies, the latter is sufficient for sustainability of public debt so that any path on which all variables grow at the same rate can be called a sustainable balanced growth path. The assumption that the primary surplus is a positive function of public debt is also motivated by empirical studies (see the papers cited in the Introduction) which present evidence that governments raise the ratio of the primary surplus to gross domestic income as the debt ratio increases.

With this assumption the analysis of our endogenous growth model produced outcomes which are different from those known in the literature. In particular, the following results could be derived.

1. If the government is a debtor, it turned out that a sustainable balanced growth path only exists if the government uses a certain part of the tax revenue for the debt service, $i_p < 1$, in case the primary surplus does not react sufficiently strong to higher public debt, i.e. for $\beta \leq \rho$. If the increase in the primary surplus is sufficiently strong as public debt rises, $\beta > \rho$, sustained growth is feasible even if no part of the tax revenue is used for the debt service, i.e. for $i_p > 1$.

2. Numerical examples demonstrated that the sustainable balanced growth path is very sensitive with respect to the fraction of the tax revenue used for public investment and with respect to the parameter $\beta$ determining the reaction of the primary surplus to a rise in public debt. This holds as concerns existence and stability of the balanced growth path. The stronger the response of the primary surplus to public debt, $\beta$, the sooner the model is stable. Further, for certain parameter constellations the model converges to stable limit cycles implying that the economy is characterized by sustained fluctuations.

3. As to growth effects of deficit financed increases in public investment, a deficit financed increase in public investment reduces the balanced growth rate for large values

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of $\beta$ because the feedback effect of the increase in public debt outweighs the initial increase in public investment. If $\beta$ is small, an increase in public investment raises the balanced growth rate but the model is unstable in this case.

4. Analyzing the transition path we could show that there is an overshooting of the growth rates over the long-run balanced growth rate following a deficit financed increase in public investment. The reason is that income effects matter as concerns the growth rates on the transition path although the income effect is irrelevant for the determination of the long-run balanced growth path.

A last remark refers to the public sector in our model. In our economy public investment is the only type of expenditure the government undertakes. Consequently, higher public debt leading to an increase in the primary surplus can only reduce productive public spending. But in reality other types of public spending, like unproductive public consumption, could be reduced, too. However, looking at real world economies it indeed seems that public investment is that type of expenditure which can be reduced most easily as public debt rises. This holds because there is no obligation for governments to invest in public infrastructure and there is no lobby group for public investment. Therefore, the decline of public investment as a result of a rising public debt is not too surprising. Empirical studies which support this view are for example Heinemann [16] who states that public debt crowds out public investment or Gong et al. [11] who find this effect for Germany and for the Netherlands.

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