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by

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Abstract

Recently, Campbell and Viceira (2002) have introduced an intertemporal framework for asset allocation problem where the interest rate and the asset price dynamics are varying with the time. This paper follows up their work and try to explain the asset allocation puzzle of Canner, Mankiw and Weil(1997). We consider the bond prices systematically by integrating the no-arbitrage bond pricing models in the intertemporal framework. We employ the method of *Dynamic Programming* to solve the intertemporal consumption and portfolio decisions numerically because usually this intertemporal decision problems cannot be solved analytically. The numerical example considers the Vasicek bond price and the Markowitz stock price. Various properties of the consumption and portfolio decisions will be demonstrated in the numerical example. Our numerical results can explain the asset allocation puzzle.

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1 Introduction

The term "strategic asset allocation" has been introduced by the book of Campbell and Viceira (2002). It refers to an asset allocation decision in a long term model where the consumption path is optimized in a time-varying investment environment. It may reflect better the real world investment opportunities. Also, academic work has been questioned since the static portfolio model of Markowitz cannot explain the *asset allocation puzzle*: Canner, Mankiw and Weil (1997) find that the financial advisors suggest more conservative investors to hold a larger proportion of bonds to stocks. However, according to the static model, the proportion of bond to stock investments of conservative investors should be as *same* as aggressive investors.

Much research has been undertaken to find a suitable model to explain this puzzle. The framework is laid out by Merton (1973) and considers long term portfolio decisions in a time-changing environment. He employs a continuous-time framework and decomposes the long-term decision into a static portfolio decision and an intertemporal hedging term. In the part of the static portfolio decision, the risky assets have the same proportion (mutual fund) whatever risk aversion investors have. Therefore, the intertemporal term should provide the explanation for the different investment proportion in the risky assets. Many recent researchers have contributed to model the intertemporal terms properly. In a finite-time framework, Kim and Omberg (1996) consider a mean-reverting market price of risk, in the work of Brennan, Schwartz and Lagnado (1997) the time-varying factors are modeled by the short-term interest rate, a long-term bond yield and dividends of a stock and Brennan and Xia (2002) account for inflation risk. Liu (2001) gives the conditions for analytical solutions for this long-term decision problem. The recent work of Munk, Sørensen and Vinther (2004) considers both inflation risk and a mean-reverting equity premium and they calibrate the model to empirical data. In an infinite-time framework, Cambpell and Viceira employ recursive utility. Their work explores the intertemporal decision problem in models with stochastic interest rates (2001) , with AR(1) equity premium (1999,2001) and labor incomes (2002) .

For bond pricing we consider in this paper *no-arbitrage* bond pricing. The reason is: the bonds of different maturities can substitute each other. Therefore, the interest rate and the bond yields of different maturities should satisfy some constraint in order to rule out arbitrage opportunity by constructing bond portfolios. The first arbitrage-free bond pricing model has been provided by Vasicek (1977) . Then, many other models followed, for example, Cox, Ingersoll and Ross (1985), Brennan and Schwartz(1979), Hull and White (1990,1994), and so on. Health, Jarrow and Morton (1992) unify all the no-arbitrage bond pricing models in their framework. Our paper will apply a tractable subclass of no-arbitrage pricing models : the *yield-factor model* of interest rate provided by Duffie and Kan (1996). It can accommodate many prominent term structure models, for example, Vasicek (1977), Cox, Ingersoll and Ross(1985), the Hull and White (1990) one-factor model and the Hull and White(1994) two-factor model.

The investment opportunities in this paper including bonds and stocks and the plan horizon is infinite. When considering bonds for an infinite horizon the following questions will arise. What we should do if the bonds mature? Which maturity of bonds should be chosen and whether the consumption and portfolio decisions depend on the bond maturities? Besides, the price dynamics of the bonds change along the time toward maturity dates. [1] So, as the time goes forwards, three things are varying: the utility of consumption is discounted, the interest rate evolves and the price dynamics are changing. As an intertemporal optimizer, the agents have to consider the interdependence due to the time between the discounting effect of the utility, the price changing effect and the evolution of the interest rate. As a result, the intertemporal decisions will depend on time in a complicated way. In this paper we will show that if a complete arbitrage-free bond market is considered, the consumption decision is independent of the bond choice. The portfolio decision depends on the bond choice but the wealth dynamics under the optimal portfolio decision are independent of the bond choice. Therefore, when a bond matures, we can continue its position with an arbitrage bond without affecting the consumption decision and the wealth dynamics under optimal portfolio decision. Also, in this case, the time has only a discounting effect.

To solve the intertemporal optimization problem we employ the *method of dynamic programming*. Numerical procedures are developed based on the discrete-time scheme. This is necessary, because even for a simple example[2] such intertemporal optimization problems cannot be solved analytically yet. The numerical procedures are implemented on several examples and the numerical performance will be checked. The focus of the numerical study is to find out how strong are the intertemporal effects and how much the risk aversion of the investors affects the long term portfolio decisions.

The remainder of the paper is organized as follows. Section 2 introduces the intertemporal model and the arbitrage-free bond pricing model in a continuous-time framework. We solve the intertemporal optimization decisions using the method of dynamic programming. A proof of the time independence by considering a complete arbitrage-free bond market is also included. Section 3 studies the discrete-time counterpart of the model in Section 2. Section 4 presents a numerical study. Section 5 concludes the paper.

[1] The volatility of bond prices decreases.

[2] See later studies in this paper.

2 Continuous-Time Framework

2.1 The Model

In our intertemporal model there is an arbitrary number of identical agents who want to maximize the expected utility of lifetime consumption. The model is a continuous-time model so the agents can make decisions at any time $0 \leq t < \infty$. The agents have initial wealth $W_0 = w$. Let W_t denote the agents' wealth at time t. The agents devote a fraction of the wealth $\psi_t, 0 \leq \psi_t \leq 1$ to consumption and the rest to buying a portfolio of assets. The utility function of consumption is assumed to be given by

$$U(\psi_t W_t) = \frac{(\psi_t W_t)^{1-\gamma}}{1-\gamma}, \tag{1}$$

for every $t \in [0, \infty)$.

For the investment opportunity there are $n+1$ assets: n of them are risky assets and the other one is the borrowing and lending. The returns of the n risky assets are described by diffusion processes

$$\frac{dP_{it}}{P_{it}} = \mu_{it} dt + \sigma_{it} dZ_{it}, \tag{2}$$

for $i = 1, 2, \cdots, n$, where $P_{it} > 0$ denotes the price of the i-th asset at time t and $(Z_{1t}, \cdots, Z_{nt})$ is a n-dimensional vector of Brownian motions with instantaneous covariance (ρ_{ij}) where $\rho_{ij} dt = \mathbf{E}[dZ_{it} dZ_{jt}]$ for $i, j = 1, \cdots, n$. The asset indexed by 0 and representing the borrowing and lending is called the *market money account*. Its return is the short term (instantaneous) interest rate r_t, so that

$$\frac{dP_{0t}}{P_{0t}} = r_t dt. \tag{3}$$

The difference between the money market account and the risky assets can be observed by comparing the return processes (2) and (3). The return r_t of the money market account is known at the current time while the return of the risky assets are exposed to uncertainty represented by the noise $\sigma_{it} dZ_{iz}$ in (2).

In the model there are state variables which are exogeneously given and affect the development of the process of the asset return. Examples of such state variables are macroeconomic factors or technology factors. We call our model a m-factor model if the number of the state variables is m. Let $X_t = (X_{1t}, \cdots, X_{mt})$ be a m-dimensional stochastic process representing the state factors. Their dynamics are assumed to follow the diffusion processes

$$dX_{it} = f_i(X_t) dt + g_i(X_t) dQ_{it}, \tag{4}$$

for $i = 1, \cdots, m$, where $f_i : \mathbb{R}^m \to \mathbb{R}$, $g_i : \mathbb{R}^m \to \mathbb{R}$, $(Q_{1t}, \cdots, Q_{mt})$ is a m-dimensional vector of Brownian motions with instantaneous covariance (ν_{ij}) where $\nu_{ij} dt = \mathbf{E}[dQ_{it} dQ_{jt}]$ for $i, j = 1, \cdots, m$. The influence of the state factors on the dynamics of the asset returns is modelled as follows. The short term riskless interest rates (3), the drift coefficients μ_{it} and the diffusion coefficients σ_{it} in the return dynamics (2) are functions of the state factors

$$\frac{dP_{0t}}{P_{0t}} = r(X_t)dt \tag{5}$$

$$\frac{dP_{it}}{P_{it}} = \mu_i(X_t, t)dt + \sigma_i(X_t, t)dZ_{it}, \tag{6}$$

where μ_i, σ_i now are functions $\mu_i : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ and $\sigma_i : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$. In addition, the drift and diffusion coefficients are allowed to be functions of t because we will consider bonds as risky assets later. Their coefficients will depend on the maturity date of the bonds. Moreover, the shocks to the state factors dQ_{it} and the shocks to the risky asset returns dZ_{it} are allowed to be correlated since they could affect each other. Let $(\eta_{ij})dt$ be the instantaneous covariance $(\eta_{ij})dt = \mathbf{E}[dQ_{it}dZ_{jt}]$ for $i = 1, \cdots, d$ and $j = 1, \cdots, n$. In short, there are time-varying state variables which change the driving force of the asset returns and the asset returns affect interactively the development of the state variables. These dynamical features of the state variables and the asset returns including their interaction is the reason why we consider *intertemporal* models instead of a static (one-period) one.

The agents live for an infinitely long time $t \in [0, \infty)$ and they decide their consumption and portfolio plans at the time $t = 0$. The portfolio decision at time t is denoted by a $n+1$-dimensional vector of real numbers $\alpha_t = (\alpha_{0t}, \alpha_{1t}, \cdots, \alpha_{nt})$ where α_{it} is the investment proportion in the i-th asset so that $\sum_{i=0}^{n} \alpha_{it} = 1$ for all $0 \leq t < \infty$. Short sales are allowed so that α_{it} can be negative. At the time $t = 0$ the agents choose *paths* of consumption ratios ψ_t and portfolio decisions α_t for the whole life $0 \leq t < \infty$ so that the expected lifetime utility is maximized. The objective function for the agents is described by

$$\max_{\psi_t, \alpha_t, 0 \leq t < \infty} \mathbf{E}\Big[\int_{t=0}^{\infty} e^{-\delta t} U(\psi_t W_t) dt\Big], \tag{7}$$

where δ is the subjective discount factor representing how the agents discount utility over time. Taking into account the asset returns (5) and (6) for a given choice of α_t and ψ_t at time t the agents' wealth evolves according to[3]

$$\frac{dW_t}{W_t} = \Big(r_t - \psi_t + \sum_{i=1}^{n} \alpha_{it}(\mu_{it} - r_t)\Big)dt + \sum_{i=1}^{n} \alpha_{it}\sigma_{it}dZ_{it}. \tag{8}$$

[3]See P.377-380 Merton(1971)

2.2 Solving the Intertemporal Decision Problem using Dynamic Programming

Let $J(t, w, x) : \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}^m \to \mathbb{R}_{++}$ be the optimized objective function at the time t, that is

$$J(t, w, x) = \max_{\psi_s, \alpha_s, s \geq t} \mathbf{E}\big[\int_t^\infty e^{-\delta s} U(\psi_s W_s) ds \big], \tag{9}$$

with initial condition $W_t = w$ and $X_t = x$. If the solution $J(t, w, x)$ exists, it should satisfy the the *Hamilton-Jacobi-Bellman*(HJB) equation [4]

$$\begin{aligned} 0 \quad = \quad & \max_{\psi_t, \alpha_t} \Big(e^{-\delta t} U(\psi_t w) dt + J_t dt + J_w \mathbf{E}[dW_t] + \sum_{i=1}^m J_{x_i} \mathbf{E}[dX_{it}] \\ & + \frac{1}{2} J_{ww} (dW_t)^2 + \sum_{i,j=1}^m \frac{1}{2} J_{x_i x_j} dX_{it} dX_{jt} + \sum_{i=1}^m J_{wx_i} dW_t dX_{it} \Big), \end{aligned} \tag{10}$$

where J_t, J_w, J_{x_i} denote the partial derivatives w.r.t t, w, x_i and $J_{ww}, J_{x_i x_j}, J_{wx_i}$ are second order partial derivatives. We shall drop the arguments of the function $J(t, w, x)$ in order to simply the notation if this does not lead to confusion. Rewriting (10) by replacing dW_t and dX_t as given by (8) and (4), using the rules of stochastic calculus [5] and cancelling the common factor dt the HJB equation (10) becomes

$$\begin{aligned} 0 \quad = \quad & \max_{\psi_t, \alpha_t} \Big\{ e^{-\delta t} U(\psi_t w) + J_t \\ & + J_w \Big[r(x) - \psi_t + \sum_{i=1}^n (\alpha_{it} \big(\mu_i(x, t) - r(x) \big) \Big] w + \sum_{i=1}^m J_{x_i} f_i(x) \\ & + \frac{1}{2} J_{ww} \big[\sum_{i=1}^n \sum_{j=1}^n \alpha_{it} \alpha_{jt} \sigma_i(x, t) \sigma_j(x, t) \rho_{ij} \big] w^2 \\ & + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m J_{x_i x_j} g_i(x) g_j(x) \nu_{ij} + \big[\sum_{j=1}^m J_{wx_j} g_j(x) \sum_{i=1}^n \alpha_{it} \sigma_i(x) \eta_{ij} \big] w \Big\}. \end{aligned} \tag{11}$$

In the following derivation we use the abbreviation $\sigma_{it} = \sigma_i(x, t)$, $\mu_{it} = \mu_i(x, t)$ for $i = 1, \cdots, n$, $r = r(x)$ and $g_i = g_i(x)$, $f_i = f_i(x)$ for $i = 1, \cdots, m$ so that the notation is less burdensome.

Let Ω be the covariance matrix of the shocks of the asset returns,

$$\Omega_t = \begin{pmatrix} \sigma_{1t}\sigma_{1t} & \cdots & \sigma_{1t}\sigma_{nt}\rho_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{nt}\sigma_{1t}\rho_{n1} & \cdots & \sigma_{nt}\sigma_{nt} \end{pmatrix}.$$

[4] The HJB equation states that the optimal lifetime utility over $[t, \infty)$ should be equal to the optimal momentum utility for a short time interval $[t, t + dt)$ plus the optimal lifetime utility over $[t+dt, \infty)$. See P.264-271 in Kamien and Schwartz (1991) for a heuristic discussion and Chapter 11 in Øksendal(2000) for a rigorous derivation.

[5] See, for example, Chap. 9 in Campbell, Lo and MacKinlay(1997)

Using the first order condition obtained from the equation (11) by differentiating w.r.t α we can solve for α to obtain

$$\begin{pmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{nt} \end{pmatrix} = -\frac{J_w}{J_{ww}\, w}\Omega_t^{-1} \begin{pmatrix} \mu_{1t} - r \\ \vdots \\ \mu_{nt} - r \end{pmatrix} - \sum_{j=1}^{m} \frac{J_{wx_j}}{J_{ww}\, w}\Omega_t^{-1} \begin{pmatrix} \sigma_{1t}\eta_{1j}g_j \\ \vdots \\ \sigma_{nt}\eta_{nj}g_j \end{pmatrix}. \tag{12}$$

If $g_j(x) \equiv 0$ for all j, then the factor are constant over the time. Then the intertemporal model is reduced to a static model and the second term in (12) is equal to zero. So, the first term is called the *static portfolio decision* and the second term is called by Merton [6] the term of *intertemporal hedging*. It is also the case when the factor noise and the asset return noise are uncorrelated.

The first order condition obtained by differentiating w.r.t ψ provides the equation

$$U'(\psi_t w) = e^{\delta t} J_w \ , \tag{13}$$

which implicitly gives the optimal consumption ratio ψ. We also know that the extreme solutions α and ψ which satisfy the F.O.C above are indeed maximizers.[7]

Using the F.O.C's (12) and (13) to rewrite (11), we obtain

$$\begin{aligned} 0 &= e^{-\delta t}U(\psi^* w) + J_t + J_w(r - \psi^*)w + \sum_{i=1}^{m} J_{x_i} f_i \\ &- \frac{1}{2}\frac{J_w^2}{J_{ww}}(\mu_t - r)'\Omega_t^{-1}(\mu_t - r) - \sum_{j=1}^{m} \frac{J_w J_{w,x_j}}{J_{ww}}(\mu_t - r)'\Omega_t^{-1}V_{jt} \\ &- \frac{1}{2}\sum_{j,k=1}^{m} \frac{J_{w,x_j}J_{w,x_k}}{J_{ww}}V_{jt}'\Omega_t^{-1}V_{kt} + \frac{1}{2}\sum_{j,k=1} J_{x_j,x_k}\, g_j g_k \nu_{jk} \ , \end{aligned} \tag{15}$$

where $\mu_{\mathbf{t}} = \begin{pmatrix} \mu_{1t} & \cdots & \mu_{nt} \end{pmatrix}'$ denotes the vector of the mean returns and

$$V_{jt} = \begin{pmatrix} \sigma_{1t}\eta_{1j}g_j \\ \vdots \\ \sigma_{nt}\eta_{nj}g_j \end{pmatrix}$$

[6] See P.876 Merton (1973).

[7] Differentiating equation (11) twice w.r.t ψ we obtain

$$e^{\delta t}U''(\psi w)w < 0,$$

because of the concavity of U. Differentiating equation (13) w.r.t w we have

$$U''(\psi w)\psi = e^{\delta t}J_{ww}. \tag{14}$$

Therefore $J_{ww} < 0$. Differentiating equation (11) twice w.r.t α we have the Hessian matrix of α is equal to $J_{ww}\Omega w^2$ which is positive definite. Therefore we know the solution of α given in (12) and ψ given in (13) are maximizers of equation (11).

denotes the covariance between the shocks of all assets and the j-th state factors. Here $'$ denotes the matrix transpose and $\psi^* = \psi^*(t, w, x)$ denotes the solution of ψ satisfying equation (13).

The equation (15) is a non-linear partial differential equation for the optimized value function $J(t, w, x)$. We saw already in (12) and (13) that $J(t, w, x)$ is needed to solve the optimal portfolio weight vector α and the consumption ratio ψ. Thus the main task at-hand is to develop ways to solve partial differential equation (15).

2.3 The Arbitrage-free Bond Pricing Model

Typical assets for investment are bonds and stocks. Bond assets give a fixed payout at a specified maturity day. [8] So they are considered as safer assets than stocks. Due to the fixed payout the dynamics of the bond returns change with the time to maturity. Let T be the maturity date and t be the current time, the drift and diffusion coefficient of the bond return process μ_i and σ_i are also functions of the time to maturity $T-t$. Under the assumption of no transaction costs, bonds can be substituted perfectly with each other. In order to exclude arbitrage possibility by creating bond portfolio we employ *arbitrage-free* bond pricing models.

Let $P(r(X_t), t, T)$ be the bond price at the time t maturing at T depending on the interest rate $r(X_t)$. For the given initial state of the factors $X_t = x$, using Itô's formula, the rate of bond return is given by [9]

$$\frac{1}{P(x,t,T)}\frac{dP(x,t,T)}{dt} = \mu_P(x,t,T)dt - \sum_{i=1}^{d} \sigma_{P,i}(x,t,T)dQ_{it}$$

where

$$\begin{aligned}
\mu_P(x,t,T) &= \frac{1}{P(x,t,T)}\Big(P_t(x,t,T) + \sum_{i=1}^{d} f_i(x)P_{x_i}(x,t,T) + \mathcal{D}P(x,t,T)\Big) \\
\sigma_{P,i}(x,t,T) &= -\frac{1}{P(x,t,T)}P_{x_i}(x,t,T)g_i(x). \qquad (16)
\end{aligned}$$

The operator $\mathcal{D}$ is defined as $\frac{1}{2}\sum_{i,j=1}^{d} g_i(x)g_j(x)\partial^2_{x_i,x_j}$. The *no-arbitrage constraint* gives a restriction between the drift and the diffusion coefficients of the bond returns. If the bond market is arbitrage-free, then there exists d functions $\lambda_i(x,t)$ for $i = 1, \cdots, d$ which are independent of the maturity date T such that

$$\mu_P(x,t,T) - r(x) = \sum_{i=1}^{d} \lambda_i(x,t)\sigma_{P,i}(x,t,T). \qquad (17)$$

[8] In our model with infinite time horizon we assume as long as a bond matures, an other bond with the same maturity duration will be introduced immediately in the opportunity set so that the number of the assets remains the same. However, it does matter the solution of ψ_t and α_t in the HJB equation (11) since the solution is only for the current time t.

[9] See Chiarella (2004).

λ_i is interpreted as the *market price of risk* for the state factor i. [10]

For our model we consider a subclass of the arbitrage-free bond models: the *yield-factor model* provided by Duffie and Kan(1996). The bond price of the yield-factor model is *exponential-affine*

$$P(x,t,T) = \exp\big(-A(T-t) - \sum_{i=1}^{d} B_i(T-t)x_i\big), \tag{18}$$

with $x = (x_1, \cdots, x_d)$.[11] The condition on the stochastic process of the factors is that $r(x)$, $f_i(x)$ for any $i = 1, \cdots, d$ and $g_i(x)g_j(x)$ for any pair $(i,j), i,j = 1, \cdots, j$ in (4) are affine in x. [12]

We assume a *complete bond market* in the model which means that the agents are allowed to invest in all bonds. If no-arbitrage constraint holds, we need only to consider d different bonds because we can derive the other bond price through the d given bonds. [13] Let T_j with $T_j > t$ be the next maturity date for the j-th bond for $j = 1, \cdots, d$. Then the return of the j-th bond can be represented by

$$\frac{dP_j(x,t,T_j)}{P_j(x,t,T_j)} = \mu_j(x,t)dt - \sum_{i=1}^{d} B_i(T_j - t)\lambda_i(x)dQ_{it}, \tag{19}$$

with the expected return $\mu_j(x,t)$

$$\mu_j(x,t) = r(x) + \sum_{i=1}^{d} \lambda_i(x,t)B_i(T_j - t)g_i(x)$$

.

2.4 Solving Intertemporal Decision Problem with Arbitrage-Free Term Structure

By considering bonds in the investment set, a question arises naturally: which bonds (which maturity date) should we choose for the model? or *whether the choice of the bonds would affect the optimal decision of the agents*? We will show in the following that if we consider a complete no-arbitrage bond market, then the choice of bonds does not affect our intertemporal decision problem.

Assume here that

A1 the market prices of risk $\lambda_i(x,t) = \lambda_i(x)$ depends only on x and

A2 the coefficients of the stock return depend also only x.

[10] It implies a synthetical portfolio with zero risk should have the same return with riskless asset. See Chiarella (2004).
[11] $A(T-t)$ and $B_i(T-t)$ depend also on the parameters in $r(x)$, $f_i(x)$ and $g_i(x)$.
[12] It means $a \cdot x + b$.
[13] See Chap. 22.10 P.352 Chiarella (2004).

$$\frac{dP_{d+1,t}}{P_{d+1,t}} = \mu_{d+1}(x)dt + \sigma_{d+1}(x)dZ_t, \tag{20}$$

Then we can show

Theorem 1 *If the assumptions A1, A2 are satisfied, the bond price is exponential affine as in (18) and the bond market is complete, then*

(i) the solution $J(t,w,x)$ for the HJB equation (15) is independent of the maturity dates of the chosen bonds.

(ii) Solving the HJB equation (15) with the investment set including the stock (20) and the bonds (19) is equivalent to solving the HJB equation with the investment set including the same stock (20) and the following assets

$$\frac{d\tilde{P}_j(x,t)}{\tilde{P}_j(x,t)} = \big(r(x) + \lambda_j(x)g_j(x)\big)dt - g_j(x)dQ_{jt}\;, \tag{21}$$

which replace the bonds in the original model.

Proof See the Appendix.

The intuition for the statement (i) is that for any other bond with maturity date $\tilde{T}$, using the completeness and the no-arbitrage of the bond market, we can generate a synthetic portfolio which has the same payout and the same price. So, any other bond set with maturity dates $\tilde{T}_1, \cdots, \tilde{T}_m$ can be generated by the original bond set $T_1, \cdots, T_m$. Then, both investment environments should be equivalent. Therefore, the optimized utility $J(t,w,x)$ should not be different.

The intuition for the statement (ii) is that by rewriting the no-arbitrage bond return (19) in the form

$$\frac{dP_j(x,t,T_j)}{P_j(x,t,T_j)} = r(x)dt + \sum_{i=1}^{m} B_i(T_j - t)\big(\lambda_i(x,t)g_i(x)dt - g_i(x)dQ_{it}\big),$$

we see the excess return of the bond with the maturity date T_j is the sum of the "factor returns" (21) with the weights $B_i(T_j - t)$ for $i = 1, \cdots, m$. Therefore the "factor assets" with the returns (21) can be considered as the generators of the bonds.

From Theorem 1 we know $J(t,w,x)$ is also the solution for the intertemporal optimization problem with the new equivalent investment set (19) and (21). Noticing that the return dynamics of the new assets are not depend on t. Then the optimization problem (9) is an "autonome" problem which means the optimization depends only the initial states but not the initial time. Therefore

Corollary 1.1 *If the assumptions in Theorem 1 are satisfied, then the objective function (9) can be transformed in*

$$\begin{aligned} J(t,w,x) &= \max_{\psi_s,\alpha_s,s\geq t} \mathbf{E}_t\big[\int_t^\infty e^{-\delta s}U(\psi_s W_s)ds\big] & (22)\\ &= e^{-\delta t}\max_{\psi_s,\alpha_s,s\geq 0} \mathbf{E}_0\big[\int_0^\infty e^{-\delta s}U(\psi_s W_s)ds\big] = e^{-\delta t}J(0,w,x), & (23)\end{aligned}$$

where the initial condition for (22) is $W_t = w$ and $X_t = x$ while the initial condition for (23) is $W_0 = w$ and $X_0 = x$.

Corollary 1.2 *If all assumptions in Theorem 1 are satisfied, then the optimal consumption is independent of the choice of bonds.*

Proof Recall that the optimal consumption has to satisfy (13). It is independent of the choice of bonds because $J(t, w, x)$ is independent of the choice of the bonds.
Q.E.D.

Corollary 1.3 *Let*

$$\mathcal{B}_t = \begin{pmatrix} B_1(T_1 - t) & \cdots & B_d(T_1 - t) \\ \vdots & \ddots & \vdots \\ B_1(T_d - t) & \cdots & B_d(T_d - t) \end{pmatrix}. \tag{24}$$

If all assumptions in Theorem 1 are satisfied, then
(i) $\mathcal{B}_t \alpha_t$ is independent of the chosen bonds where α_t is the optimal portfolio decision.
(ii) The wealth dynamics (8) under the optimal portfolio decision is independent of the bond choice. Furthermore, we can decompose the wealth change (8) into three parts: the part contributed by the stock market, by the fixed income market (including the bonds and the money) and by the consumption. All three parts of the wealth changes are independent on the bond choice.

Proof See the Appendix.

Following the discussion above we seek the solution $J(t, w, x)$ of the form [14]

$$J(t, w, x) = e^{-\delta t} H(x)^\gamma U(w) \tag{25}$$

where U is the utility function defined in (1). Then the optimal consumption ratio is solved as

$$\psi^*(t, w, x) = H(x)^{-1} \tag{26}$$

and it depends only on the initial state x. The optimal portfolio weight becomes

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \mathcal{B}_t^{-1} & 0 \\ & \\ 0 & 1 \end{pmatrix} \left(\frac{1}{\gamma} \tilde{\Omega}^{-1}(\tilde{\mu} - r) + \sum_{j=1}^{d} \frac{H_{x_j}}{H} \tilde{\Omega}^{-1} \tilde{V}_j \right), \tag{27}$$

where $\tilde{\mu}, \tilde{\Omega}$ and $\tilde{V}_i$ are defined in the Appendix. [15]

[14] For the product form $H(x)^\gamma U(w)$ of the solution form see later the discussion on the discrete-time version of the model.

[15] The optimal portfolio can be found in (65). The definitions of $\tilde{\mu}, \tilde{\Omega}$ and $\tilde{V}_i$ are founded in (61,) (62) and (63).

The HJB equation (15) reduces to

$$
\begin{aligned}
0 &= \frac{\gamma}{H} + \big(-\delta + r(1-\gamma) + \frac{1-\gamma}{2\gamma}(\tilde{\mu}-r)'\tilde{\Omega}^{-1}(\tilde{\mu}-r)\big) \\
&+ \sum_{i=1}^{m} \frac{H_{x_i}}{H}\big(\gamma f_i + (1-\gamma)(\tilde{\mu}-r)'\tilde{\Omega}^{-1}\tilde{V}_i\big) \\
&+ \frac{1}{2}\sum_{i,j=1}^{m} \frac{H_{x_i x_j}}{H} g_i g_j \nu_{ij} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m} \frac{H_{x_i}H_{x_j}}{H^2}\big(\tilde{V}_i'\tilde{\Omega}^{-1}\tilde{V}_j - g_i g_j \nu_{ij}\big)
\end{aligned} \tag{28}
$$

and the problem now reduces to solving for H. At most cases, H cannot be solved analytically.[16] Later we develop numerical algorithm to solve it.

[16]It can be solved for finite plan horizon, see Liu (2001) , Kim and Omberg (1996).

3 Discrete-Time Models and the Iteration Method

3.1 Discrete-Time Version of the Continuous-Time Model

We consider here one discrete-time approximation for the continuous-time intertemporal consumption-portfolio problem described in the Section 2.1. The agents can in the discrete-time model consume and make portfolio decisions only at the given discrete time points $t_0, t_1, t_2, \cdots$. We assume in this paper that the time points are equi-distant $t_d = d * h$ with step size $h > 0$ and $d = 0, 1, 2, \cdots$. The discrete-time aspect can help to understand intuitively properties of the optimal decisions of the continuous-time model. Moreover, it lays the process of the numerical method for solving the optimal decisions later.

In the discrete-time model, the dynamics of the underlying factors (4) are approximated by the *Euler method*

$$X_{t+\Delta t} - X_t = f(X_t)\Delta t + g(X_t)(Q_{t+\Delta t} - Q_t). \tag{29}$$

[17] The price processes are approximated by

$$\frac{P_0(t_{d+1}) - P_0(t_d)}{P_0(t_d)} = r(X(t_d))h\,, \tag{30}$$

$$\frac{P_i(t_{d+1}) - P_i(t_d)}{P_i(t_d)} = \mu_i(X(t_d), t_d)h + \sigma_i(X(t_d), t_d)\Delta Z_i(t_{d+1}) \tag{31}$$

for $i = 1, \cdots, n$, where $\Delta Z_i(t_{d+1}) = Z_i(t_{d+1}) - Z_i(t_d)$ is the increment of the Brownian motion Z_i given in (6).

The agents decide the consumption ratio and portfolio *right after* the prices $P_i(t_d)$ are announced. The consumption ratio and portfolio weights decided at t_d are denoted by $\psi(t_d)$ and $\alpha_i(t_d)$ for $i = 0, \cdots, n$ and $\sum_{i=0}^{n} \alpha_i(t_d) = 1$.

The agents maximize the lifetime expected utility of consumption. The current time is 0. The agents can choose the level of consumption and the portfolio weights after a specified time t_k with given initial wealth $W_{t_k} = w$, initial state $X_{t_k} = x$. The mathematical representation of the agents' decision problem is given by

$$\overline{J}(t_k, w, x) := \max_{\psi(t_d), \alpha(t_d), d \geq k} \mathbf{E}_{t_k}\left[\sum_{d=k}^{\infty} e^{-\delta t_d} U(\psi(t_d)W(t_d))\right]. \tag{32}$$

The conditional expectation E_{t_k} represents the expectation of the wealth development after t_k given $W_{t_k} = w$ and $X_{t_k} = x$.

The wealth dynamics in the discrete-time model is described by [18]

$$W(t_{d+1}) = W(t_d)(1 - \psi(t_d)h)\big(1 + \sum_{i=0}^{n} \alpha_i(t_d)\frac{P_i(t_{d+1}) - P_i(t_d)}{P_i(t_d)}\big). \tag{33}$$

[17] Alternatively, we can employ higher order approximation method, for example, the Milstein method, see Kloeden and Platen(1994) .

[18] See the Appendix.

The wealth change is equal to the weighted sum of each asset return $\Delta P/P$ times the wealth after consumption. Placing the asset returns (31) into (33), we see that the wealth change is a function of the initial wealth, consumption ratio, portfolio weights, current states, current time and the shocks in this period. So we denote the wealth change dynamics by

$$\begin{aligned} W(t_{d+1}) &=: \varphi_W\big(W(t_d), \psi(t_d), \alpha(t_d), X(t_d), \Delta Z(t_d), t_d\big) && (34)\\ &= W(t_d)(1-\psi(t_d)h)\Pi(\alpha(t_d), X(t_d), \Delta Z(t_{d+1}), t_d), && (35)\end{aligned}$$

where

$$\begin{aligned} &\Pi(\alpha, x, \Delta Z, t) && (36)\\ = \; & 1 + r(x)h + \sum_{i=1}^{n} \alpha_i \Big[(\mu_i(x,t) - r(x))h + \sigma_i(x,t)\Delta Z_i\Big]. \end{aligned}$$

Π can be interpreted as the portfolio return for one period. It is independent of the consumption decision and the wealth level.

3.2 Solving the Discrete-Time Optimization Problem by Iteration

The Bellman equation for $\overline{J}(t, w, x)$ in the discrete-time version is given by

$$\begin{aligned} \overline{J}(t,w,x) &= \max_{\psi_t, \alpha_t} \big\{e^{-\delta t} U(\psi_t w)h + \mathbf{E}_t[\overline{J}(t+h, W_{t+h}, X_{t+h})]\big\} && (37)\\ &= \max_{\psi_t, \alpha_t} \big\{e^{-\delta t} U(\psi_t w)h + \mathbf{E}_t[\overline{J}\Big(t+h, \varphi_W(w, \psi_t, \alpha_t, x, \Delta Z_{t+h}, t), \varphi_X(x, \Delta Q_{t+h})\Big)], \end{aligned}$$

where $w = W_t$, $x = X_t$ are initial states. Let $\mathcal{T}$ be defined as the operator on $\overline{J}(t, w, x)$ such that

$$\begin{aligned} &\mathcal{T}(\overline{J})(t,w,x) && (38)\\ = \; & \max_{\psi_t, \alpha_t} \Big\{ e^{-\delta t} U(\psi_t w)h + \mathbf{E}_t[\overline{J}\Big(t+h, \varphi_W(w, \psi_t, \alpha_t, x, \Delta Z_{t+h}, t), \varphi_X(x, \Delta Q_{t+h})\Big)]\Big\}. \end{aligned}$$

Then the solution of (37) is the fixed point of the operator $\mathcal{T}$

$$\mathcal{T}(\overline{J})(t,w,x) = \overline{J}(t,w,x). \qquad (39)$$

We can start with some function $\overline{J}_0$ and apply the operator iteratively

$$\overline{J}_k(t,w,x) := \overbrace{\mathcal{T} \circ \cdots \circ \mathcal{T}}^{k\text{-times}} \circ \overline{J}_0(t,w,x) \qquad (40)$$

denotes the k-th step iteration.

Under certain conditions[19] the solution of (40) exists and is unique and $\lim_{k\to\infty} \overline{J}_k(t, w, x)$ converges to this solution.[20] We call this solution method

[19]The conditions are (i)f, g in (29), r in (30), μ_i and σ_i in (31) are bounded and Lipschtz-continuous $|F(x_1) - F(x_2)| < L_F|x_1 - x_2|$, (ii) U is bounded and Hölder continuous $|U(x_1) - U(x_2)| < L_U|x_1 - x_2|^\theta$ (iii) $h < 1/\delta$.

[20]See Camilli and Falcone(1995).

the *iteration method.*

3.3 No-Arbitrage Term Structure in Discrete-Time Models

Applying the Euler approximation method to the bond price dynamics (19) we obtain

$$\begin{aligned} & \frac{P(x,t+h,T)-P(x,t,T)}{P(x,t,T)} \qquad (41) \\ = & \Big(r(x)+\sum_{i=1}^{m}\lambda_i(x)B_i(T-t)g_i(x)\Big)h-\sum_{i=1}^{m}B_i(T-t)g_i(x)\Delta Q_{i,t+h}. \end{aligned}$$

These approximate bond returns satisfy the no-arbitrage constraint (17) and are arbitrage-free.[21] So, analog to the continuous time framework under the assumption of a complete arbitrage-free bond market, we consider a new investment set including the same stock and m assets with the following return

$$\frac{\tilde{P}_j(x,t+h)-\tilde{P}_j(x,t)}{\tilde{P}_j(x,t)}=\big(r(x)+\lambda_j(x)g_j(x)\big)h-g_j(x)\Delta Q_{j,t+h}\ , \qquad (42)$$

for $j=1,\cdots,m$ who replace the bonds.

The new investment set provides the same return as the original one because

$$\begin{aligned} \Pi(\alpha,x,\Delta Z,t) &= 1+r(x)h+\sum_{i=1}^{m}\alpha_i\sum_{j=1}^{m}B_i(T_j-t)\Big(\lambda_j(x)g_j(x)h-g_j(x)\Delta Q_{j,t+h}\Big) \\ &\quad +\alpha_{m+1}\Big((\mu_s(x)-r(x))h+\sigma_s(x)\Delta Z_{t+h}\Big) \qquad (43) \\ &= 1+r(x)h+\sum_{j=1}^{m}\tilde{\alpha}_j\Big(\lambda_j(x)g_j(x)h-g_j(x)\Delta Q_{j,t+h}\Big) \\ &\quad +\alpha_{m+1}\Big((\mu_s(x)-r(x))h+\sigma_s(x)\Delta Z_{t+h}\Big) \\ &=: \tilde{\Pi}(\tilde{\alpha},x,\Delta Z), \qquad (44) \end{aligned}$$

where

$$\begin{aligned} \tilde{\alpha}_j &= \sum_{i=1}^{m}B_i(T_j-t)\alpha_i \quad \text{for } j=1,\cdots,m \qquad (45) \\ \tilde{\alpha}_{m+1,t} &= \alpha_{m+1,t}. \qquad (46) \end{aligned}$$

We can interpret $\tilde{\alpha}_j$ as the investment weight in the j-factor-asset for the new investment set.

[21] See the footnote for (17).

3.4 Separation Theorem

For the iteration (40) we start the iteration with the function

$$\overline{J}_0(w,x) = e^{-\delta t}U(w). \tag{47}$$

We claim

Theorem 2 (Separation Theorem) *If the arbitrage-free and complete bond market is considered and the initial function for the iteration is given above, then for every iteration step $k = 0, 1, \cdots$ $\overline{J}_k$ has the factorized form:*

$$\overline{J}_k(t,w,x) = e^{-\delta t}U(w)S_k(x), \tag{48}$$

where U is defined in (1).

Remark The optimized utility function $J(t,w,x)$ can be decomposed into three multiple parts.

Proof
The proof is obtained by the mathematical induction. Assume that at the k-th step $\overline{J}_k(t,w,x)$ has the factorized form

$$\overline{J}_k(t,w,x) = e^{-\delta t}U(w)S_k(x), \tag{49}$$

then at the $k+1$-st step, for $0 < \gamma < 1$:

$$\begin{aligned}
& \overline{J}_{k+1}(t,w,x) \qquad (50)\\
= \;& \max_{\psi,\alpha}\Big\{e^{-\delta t}U(\psi w)h + e^{-\delta(t+h)}\mathbf{E}_t[U\big(\varphi_W(w_t,\alpha,\psi,x,\Delta Z)\big)S_k\big(\varphi_X(x,\Delta Q)\big)]\Big\}\\
= \;& e^{-\delta t}U(w)\max_{\psi,\alpha}\Big\{\psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma}\mathbf{E}_t[\Pi(\alpha,x,\Delta Z,t)^{1-\gamma}S_k\big(\varphi_X(x,\Delta Q)\big)]\Big\}\\
= \;& e^{-\delta t}U(w)\max_{\psi}\Big\{\psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma}\max_{\alpha}\mathbf{E}_t[\Pi(\alpha,x,\Delta Z,t)^{1-\gamma}S_k\big(\varphi_X(x,\Delta Q)\big)]\Big\}\\
= \;& e^{-\delta t}U(w)\max_{\psi}\Big\{\psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma}\max_{\tilde{\alpha}}\mathbf{E}_t[\tilde{\Pi}(\tilde{\alpha},x,\Delta Z)^{1-\gamma}S_k\big(\varphi_X(x,\Delta Q)\big)]\Big\}.
\end{aligned}$$

For $\gamma > 1$, we change all max with min because $U(w) < 0$.

The second equality is due to the property of the utility function (1)

$$U(\psi w) = \psi^{1-\gamma}U(w)$$

and the wealth dynamics (35).

The intuition for the third equality is that the consumption decision ψ does not appear in the term $\mathbf{E}_t[\Pi(\alpha,x,\Delta Z,t)^{1-\gamma}S_k\big(\varphi_X(x,\Delta Q)\big)]$. The precise proof is provided in the Appendix. [22]

[22] Theorem 3 [Separation Theorem for the agents'Decisions].

The fourth equality is based on the algebraic transformation of the portfolio return (44). Because of the equivalence $\Pi(\alpha, x, \Delta Z, t) \equiv \tilde{\Pi}(\tilde{\alpha}, x, \Delta Z)$, then we can have

$$\begin{aligned}&\max_{\alpha} \mathbf{E}_t[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)]\\ &\equiv \max_{\tilde{\alpha}} \mathbf{E}_t[\tilde{\Pi}(\tilde{\alpha}, x, \Delta Z)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)]\end{aligned}$$

and the maximization solutions α^* and $\tilde{\alpha}^*$ are related with each other by the one-to-one relationship (45). The purpose of this step is that we can rewrite $\overline{J}_{k+1}(t, w, x)$ into the separable form

$$\overline{J}_{k+1}(t, w, x) = e^{-\delta t} U(w) S_{k+1}(x)$$

where

$$S_{k+1}(x) = \max_{\psi} \left\{ \psi^{1-\gamma} h + e^{-\delta h} (1 - \psi h)^{1-\gamma} \max_{\tilde{\alpha}} \mathbf{E}_t[\tilde{\Pi}(\tilde{\alpha}, x, \Delta Z)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)] \right\} \tag{51}$$

is not a function of t.

For $\gamma > 1$ the iteration $S_{k+1}(x)$ is defined analogously

$$S_{k+1}(x) = \min_{\psi} \left\{ \psi^{1-\gamma} h + e^{-\delta h} (1 - \psi h)^{1-\gamma} \min_{\tilde{\alpha}} \mathbf{E}_t[\tilde{\Pi}(\tilde{\alpha}, x, \Delta Z)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)] \right\}. \tag{52}$$

Q.E.D.

Remark 1 From the proof above, we saw that the time schedule dependence of the wealth dynamics does not affect the maximization result. However, it indeed affects the investment weights on the bonds. To see that, we solve at first the optimization problem with the new investment set and obtain the maximizer $\tilde{\alpha}^*$. Then the optimal portfolio α^* of the original model is obtained as

$$\begin{pmatrix} \alpha^*_{1t} \\ \alpha^*_{2t} \\ \vdots \\ \alpha^*_{d,t} \end{pmatrix} \equiv \begin{pmatrix} B_1(T_1 - t) & B_1(T_2 - t) & \cdots & B_1(T_d - t) \\ B_2(T_1 - t) & B_2(T_2 - t) & \cdots & B_2(T_d - t) \\ \vdots & \vdots & \ddots & \vdots \\ B_d(T_1 - t) & B_d(T_2 - t) & \cdots & B_d(T_d - t) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\alpha}^*_{1t} \\ \tilde{\alpha}^*_{2t} \\ \vdots \\ \tilde{\alpha}^*_{d,t} \end{pmatrix}$$

For a different set of the maturity dates we will have a different optimal choice for bond investment.

Remark 2 If the limit of $\lim_{k\to\infty} S_k(x) = S(x)$ exists and is unique, then $S(x)$ is the fixed point of the iteration (51) for $0 < \gamma < 1$ (or (52) for $\gamma > 1$) and $J(t, w, x) = e^{-\delta t} U(w) S(x)$ is the solution of the Bellman equation (37). The solution $S(x)$ is independent of the choice of the initial function (47).

4 Numerical Study

Recall that the objective is to solve the optimal portfolio decision α_t and ψ_t satisfying (12) and (13) and the main problem is that $J(t,x,w)$ cannot be solved analytically. In this section we develop computational algorithm based on the iteration method described in Section 3.2. The focuses of this numerical study is to explain how strong the intertemporal hedging term in (12) is and how the parameter of the risk aversion, γ, affects the portfolio decision α_t.

Using Separation Theorem 1, $\overline{J}_k(t,w,x)$ of the iteration method can be decomposed into the multiple form $\overline{J}_k(t,w,x) = e^{-\delta t}U(w)S_k(x)$. The iteration process for $S_k(x)$ is described in (51). It is advantageous to use the multiple form because we can reduce the iteration by two dimensions. For example, if the underlying factor is one-dimensional, we will implement one-dimensional iteration for $S(x)$ instead of three-dimensional problem for $\overline{J}(t,w,x)$.

The numerical solutions ψ_t and α_t will be compared with the solutions in the continuous-time model (27) and (26).

4.1 Numerical Implementation

The Brownian Motion increments ΔZ_i and ΔQ_i in Section 3.2 are approximated by the random walks.[23] The process the underlying factors (29) is constrained on a compact set. Although the underlying factors are stochastic processes, for our numerical examples later we can find a compact set large enough such that the stochastic processes will "almost" always stay within this set. Then, this constraint is still acceptable.

For each iteration we have to know $S_k(x)$ on the compact set. We use the standard numerical method: taking a grid on the compact set of the state variables, evaluating $S_k(x)$ evaluated on the grid points and approximating other values by the bilinear form, for details see Grüne (1997) and Grüne (2001).

The numerical procedure is programmed with the programming language "GAUSS".[24] To find the maximizer in (51) the BFGS method[25] is used.

4.2 Examples

Here we consider the investment set which includes

[23]The one-dimensional random walk is

$$P\big(B(t_{d+1}) - B(t_d) = \sqrt{h}\big) = P\big(B(t_{d+1}) - B(t_d) = -\sqrt{h}\big) = \frac{1}{2}.$$

For n-dimensional Brownian Motion increments with given correlation we take the corresponding linear combination of n independent Random Walks to obtain the specified correlations.

[24]Commercial software from Aptech Systems (www.aptech.com).

[25]See "www.aptech.com/papers/qnewton.pdf".

- money market account with the return process[26]

$$\frac{dP_{0t}}{P_{0t}} = dr_t = \kappa(\theta - r_t)dt + g_0 dQ_t, \tag{53}$$

  where κ,θ and g_0 are positive constants. This process is also called a *mean-reverting process.* [27]

- A bond matures at T for $T > t$ with return[28]

$$\frac{dP_{1t}}{P_{1t}} = \big(r_t + \lambda B(T-t)g_0\big)dt - B(T-t)g_0 dQ_t, \tag{54}$$

  where λ represents the market price of interest rate risk and $B(\tau)$ is the function

$$B(\tau) = \frac{1 - \exp(-\kappa\tau)}{\kappa}.$$

- A stock with return

$$\frac{dP_{2t}}{P_{2t}} = \mu_s dt + \sigma_s dZ_t. \tag{55}$$

The correlation between the two Brownian motions dQ_t and dZ_t is η.

The control variable is the consumption proportion ψ_t, the investment proportion of the bond α_{1t} and the stock α_{2t}. The state variable is the interest rate r_t.

The parameters employed in the simulation example are:

for the interest rate:	$\theta = 0.04$, $g_0 = 0.0016$, $\kappa = 0.46$,
for the bond:	$\lambda = 0.005$, $T = 1$,
for the stock:	$\mu_s = 0.045$, $\sigma_s = 0.25$,
the correlation:	$\eta = -0.6$, 0 and 0.6,
the risk aversion	$\gamma = 0.1, 0.5, 0.9$.

And the parameters taken for numerical implementation are:

the current time: $t = 0$,
time discretization: $h = 0.1$
the compact set for the state variable: $r_t \in [0.01, 0.07]$,
the number of the grid points: $= 61$,
the width of the grid cells $\Delta r = 0.001$.

Figure 1 shows a typical path of the interest rate process. The limit distribution of this process is a normal distribution with mean 0.04 and standard deviation 0.00167. Then the probability to move out of the given compact set is $3.72 * 10^{-72}$.

We consider two examples in our numerical experiments.

[26] See Vasicek (1977)
[27] θ is the "mean" for the interest rate r_t. κ represents the reversion speed to the mean θ.
[28] See Vasicek (1977)

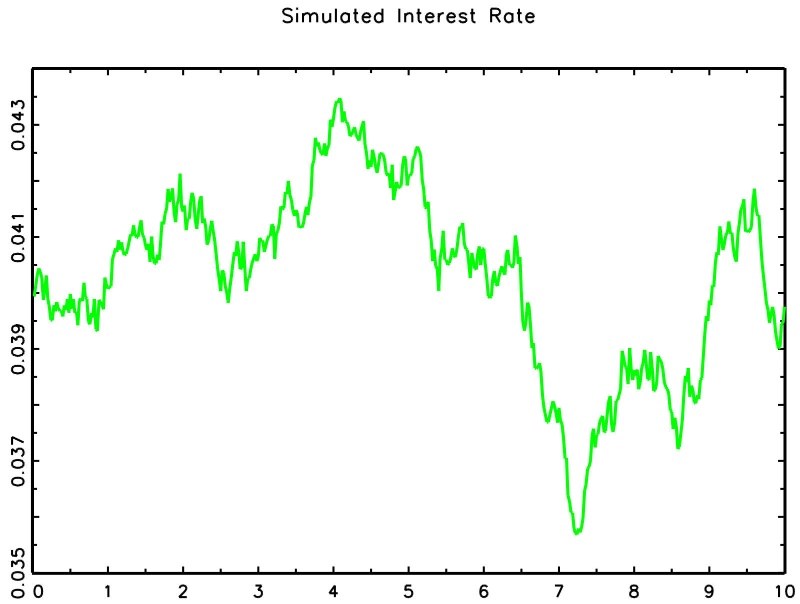

Figure 1: One Realization of the Vasicek Interest Rate

4.2.1 An Example excluding Bonds

The bond is not included in the investment set at first. The control variable is ψ the consumption ratio and α the investment weight on the stock so that $1-\alpha$ is the investment on money market account. The state variable is the instantaneous interest rate.

Using the result of (27), the optimal stock investment is given by

$$\alpha = \frac{1}{\gamma}\frac{\mu_s - r_t}{\sigma_s^2} + \frac{H'(r)}{H(r)}\frac{g_0}{\sigma_s}\eta. \tag{56}$$

The first term is the static portfolio and the second term is the intertemporal hegding term. In this case the function $H(r)$ cannot be solved analytically.

If $\eta = 0$, then the optimized stock investment is equal to the static stock investment. We employ this fact to check the performance of our numerical procedures. We compare the numerical and the static optimal portfolio choices and summarize their average absolute errors in the following:

$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.9$
0.0409	0.0014	0.0044

The error for $\gamma = 0.1$ is larger than those for $\gamma = 0.5, 0.9$. This is because the more risk-friendly agents also take larger positions on the risky assets.

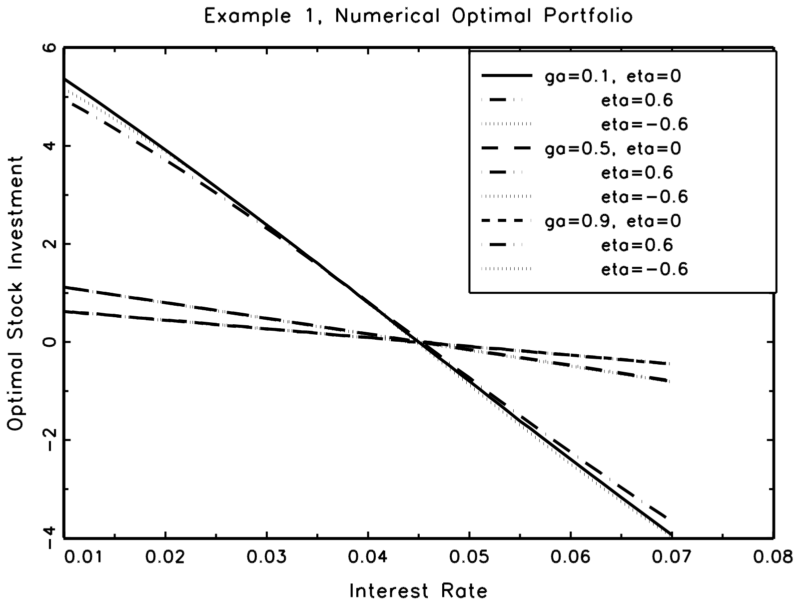

Figure 2: Numerical Optimal Portfolio Choice

In Figure 2 we observe the numerical solutions for the optimal stock investment. As we can observe for the same risk aversion γ, the portfolios are close for different correlation η. This means that the intertemporal hedging terms do not contribute much to the portfolio choice. We analyze the intertemporal effects further. In Figures 3 and 4 the curves denoted "num"(numerical) plot the difference between the numerical solution and the static stock investment and "theo"(theoretical) plot the term $\frac{H'(r)}{H(r)} \frac{g_0}{\sigma_s} \eta$ using the numerical results for $H(r)$. The scale of the intertemporal effects depend on the slope of $H(x)$. In Figures 5 and 6 we see the values of $H(r)$. We observe in Figure 5 that the function $H(r)$ is increasing with r while in Figure 6 for $\gamma = 0.1$ the function $H(r)$ is decreasing for small r and increasing for large r. The economic explanation is following. Usually, the higher interest rate r increases utility because r represents a "minimal" return – the return just on the money market and without investment strategies. However, the agents with small risk aversion $\gamma = 0.1$ are willing to take more risk and construct extreme portfolio for potential higher expected profit. When r is small, the opportunity cost is low, they can have high utility for high expected return from risky assets. Therefore, the utility $H(r)$ decreases with r for small r. From Figures 5 and 6 we obtain information about the intertemporal effects from the slope of $H(r)$. The more aggressive the risk attitude of the agents (smaller γ), the stronger is the intertemporal effect. For small r and for $\gamma = 0.1$, the intertemporal effect is opposite to the effect for large r due to the negative value of $H'(r)$. For example for $\eta = 0.6$, it is negative when r small and is positive when r is large.

The small size of the intertemporal terms in this example is due to the small standard deviation ratio $\frac{g}{\sigma_0} = 0.0064$. Thus, the intertemporal terms would be stronger when interest rates are more volatile.

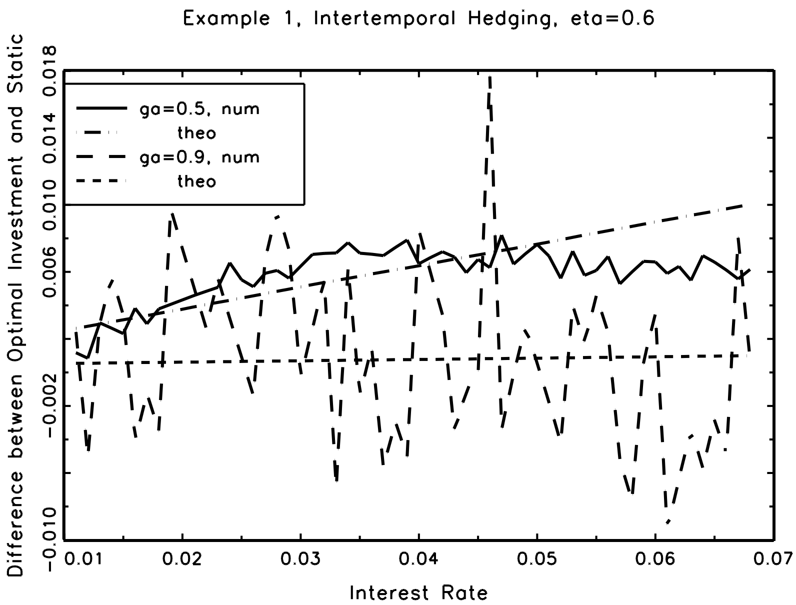

Figure 3: Intertemporal Hedging Terms for $\gamma = 0.5, 0.9$

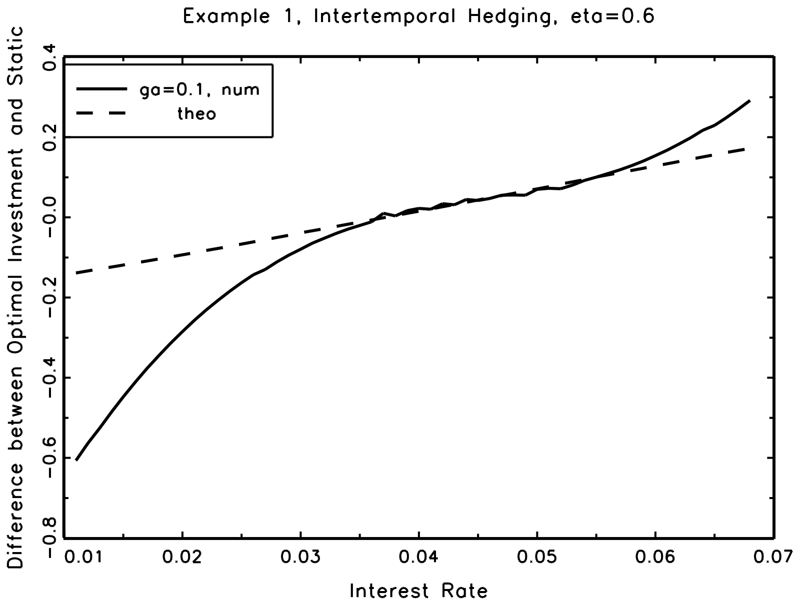

Figure 4: Intertemporal Hedging Terms for $\gamma = 0.1$

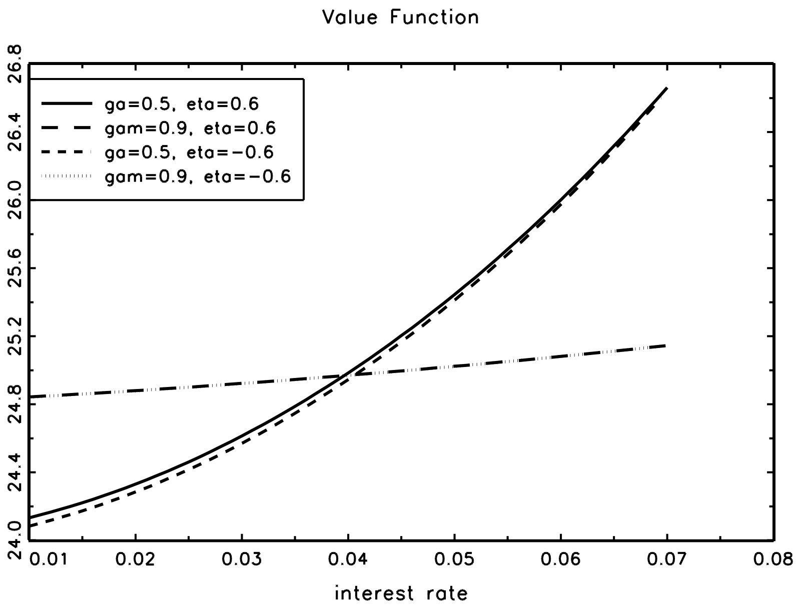

Figure 5: Value of $H(r)$ for $\gamma = 0.5, 0.9$

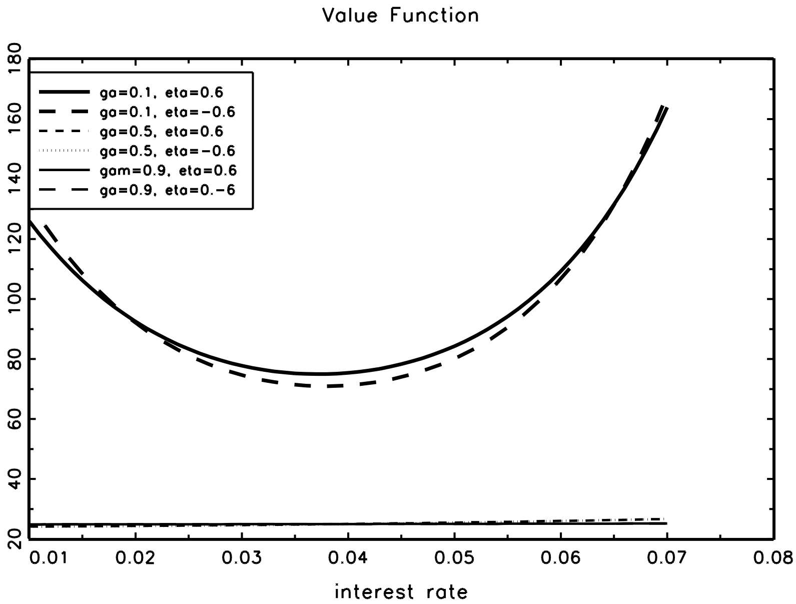

Figure 6: Value of $H(r)$

4.2.2 The Example including the Bond

All three assets, money market account, the bond and the stock are now considered. There is one state variable: the interest rate and there are three control variables: ψ_t, the investment weights on the bond α_{1t} and the stock α_{2t}. The optimal portfolio weights α_{1t}, α_{2t} in the continuous-time model, using the result (27), are

$$\begin{pmatrix} \alpha_{1t} \\ \alpha_{2t} \end{pmatrix} = \begin{pmatrix} \frac{1}{B(T-t)} & 0 \\ 0 & 1 \end{pmatrix} \left(\frac{1}{\gamma} \tilde{\Omega}^{-1} (\tilde{\mu} - r) + \frac{H'(r)}{H(r)} \tilde{\Omega}^{-1} \tilde{V} \right) \tag{57}$$

where

$$\begin{aligned} \tilde{\Omega} &= \begin{pmatrix} g_0^2 & -g_0 \sigma_s \eta \\ -g_0 \sigma_s \eta & \sigma_s^2 \end{pmatrix} \\ (\tilde{\mu} - r) &= \begin{pmatrix} \lambda g_0 \\ \mu_s - r_t \end{pmatrix} \\ \tilde{V} &= \begin{pmatrix} -g_0^2 \\ \sigma_s \eta g_0 \end{pmatrix}. \end{aligned}$$

Notice that

$$\tilde{\Omega}^{-1} \tilde{V} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

due to the perfect correlation of the bond return noise and interest rate noise. So the second term of the equation (57) reduces to

$$\begin{pmatrix} \alpha_{1t} \\ \alpha_{2t} \end{pmatrix} = \frac{1}{\gamma(1-\eta^2)} \begin{pmatrix} \frac{\eta(\mu_s - r_t) + \sigma_s \lambda}{g_0 \sigma_s B(T-t)} \\ \frac{\mu_s - r_t + \sigma_s \eta \lambda}{\sigma_s^2} \end{pmatrix} - \begin{pmatrix} \frac{1}{B(T-t)} \frac{H'(r)}{H(r)} \\ 0 \end{pmatrix}. \tag{58}$$

The noticeable point here is, if the no-arbitrage bond market is considered, then the intertemporal hedging for the stock investment is equal to zero, whatever the correlations of the stock return noise and the interest rate noise are. This is the insight of the *three-fund theorem* of Merton (1990) [29] that, if there is an asset perfectly correlated with the shocks of the factors, then the intertemporal impact of the factors on the other assets are absorbed by the perfect correlation.

This point is very convenient for checking the performance of the numerical result because the stock investment is equal to the static stock investment which depends only on the known parameters and can be calculated easily. The average errors between the numerical and theoretical stock investment proportion are listed in Table 1. We can see again that the numerical errors for more risk-friendly agents (with $\gamma = 0.1$) are larger than the others. In Figures 7, 9 and 11 we can compare the numerical errors to their levels. The numerical result is quite satisfactory.

	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.9$
$\eta = 0.0$	0.0407	0.0022	0.0012
$\eta = 0.6$	0.1088	0.0039	0.0042
$\eta = -0.6$	0.0988	0.0039	0.0407

Table 1: Average Error of Example 2 for Stock Investment

In Figures 8, 10 and 12 numerical results of the bond investment are displayed. The positions of bond holding are quite large. We can observe that the sizes of static bond choice are already large. This can be explained by the small g_0 in the first row of the formula (58).

The intertemporal effect is not significant for $\eta = 0.6$ and -0.6. The intertemporal effect is only significant for $\eta = 0$ and $\gamma = 0.1$. Following (58) the intertemporal effect is represented by the term $-\frac{H'(r)}{H(r)B(T-t)}$. In Figure 8 we can observe that the numerical result of the term $-\frac{H'(r)}{H(r)B(T-t)}$ can explain the numerical intertemporal effect for the bond (the numerical optimal choice minus the static portfolio choice) quite well. We can observe in Figures 8, 10 and 12 that the intertemporal effect for $\gamma = 0.1$ is positive for small r and negative for large r. This can be explained by the form of the value function $H(r)$ illustrated in Figure 14. For small r, $H(r)$ is decreasing, so $-H'(r)$ is positive.

[29] in Chap.15.7.

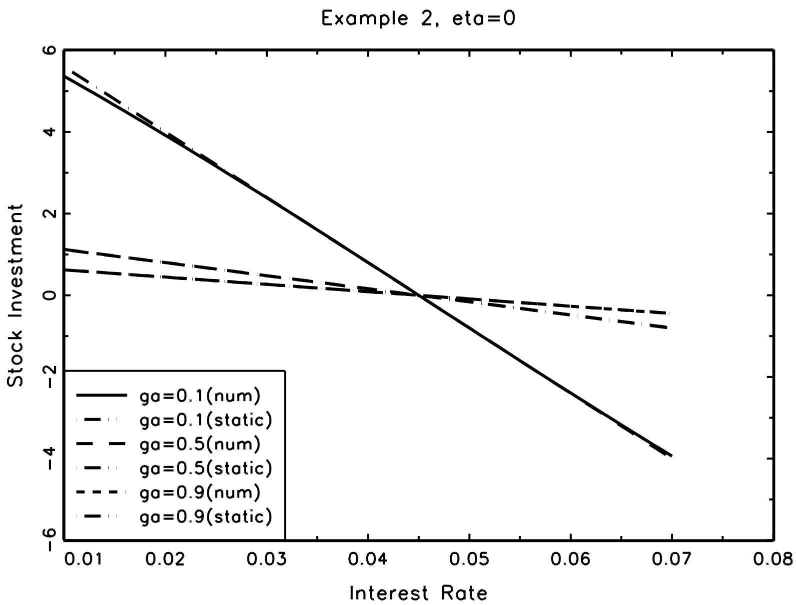

Figure 7: Portfolio Choice for Stock, $\eta = 0$

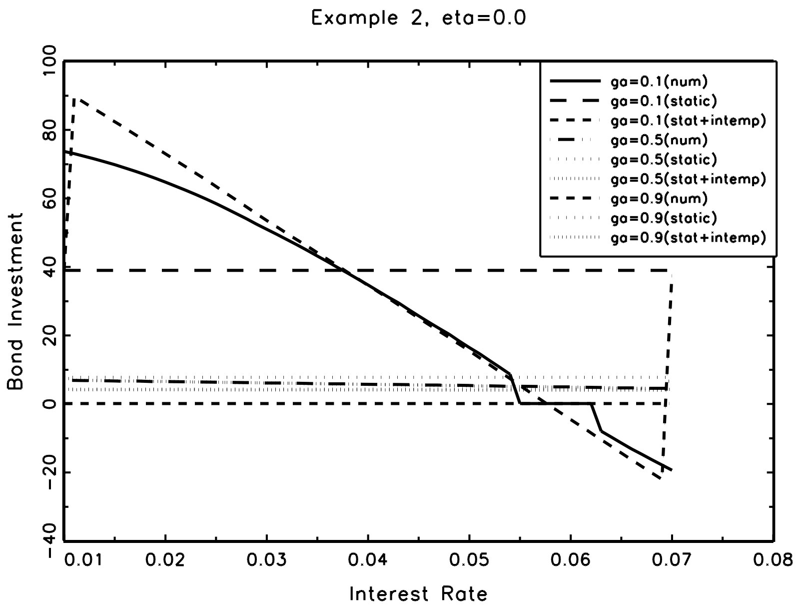

Figure 8: Portfolio Choice for Bond, $\eta = 0$

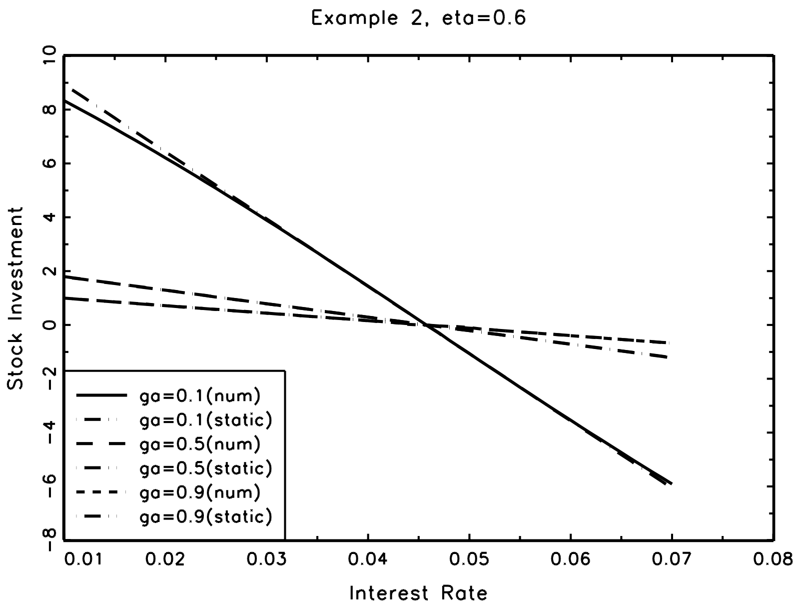

Figure 9: Portfolio Choice for Stock, $\eta = 0.6$

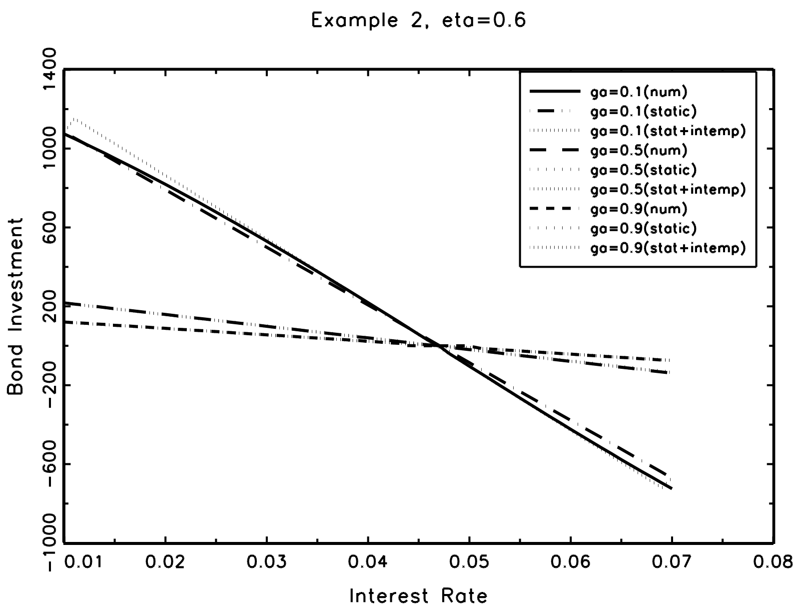

Figure 10: Portfolio Choice for Bond, $\eta = 0.6$

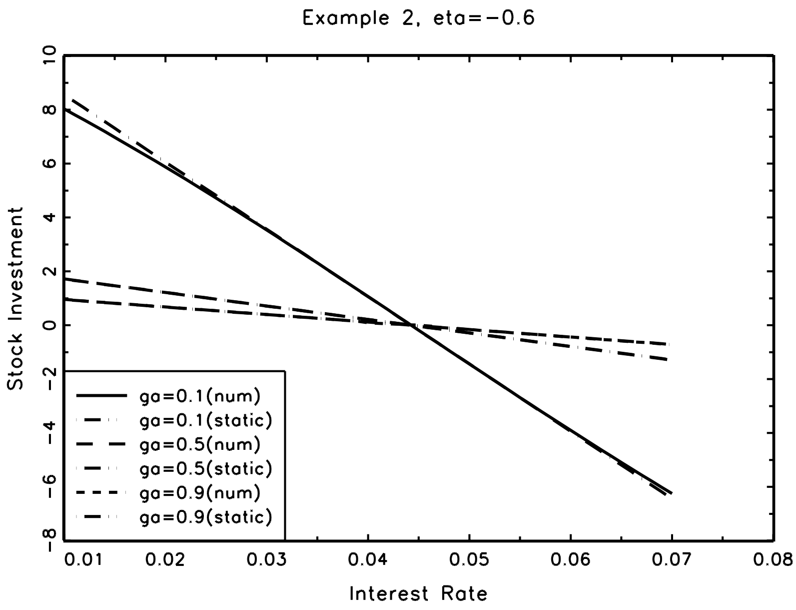

Figure 11: Portfolio Choice for Stock, $\eta = -0.6$

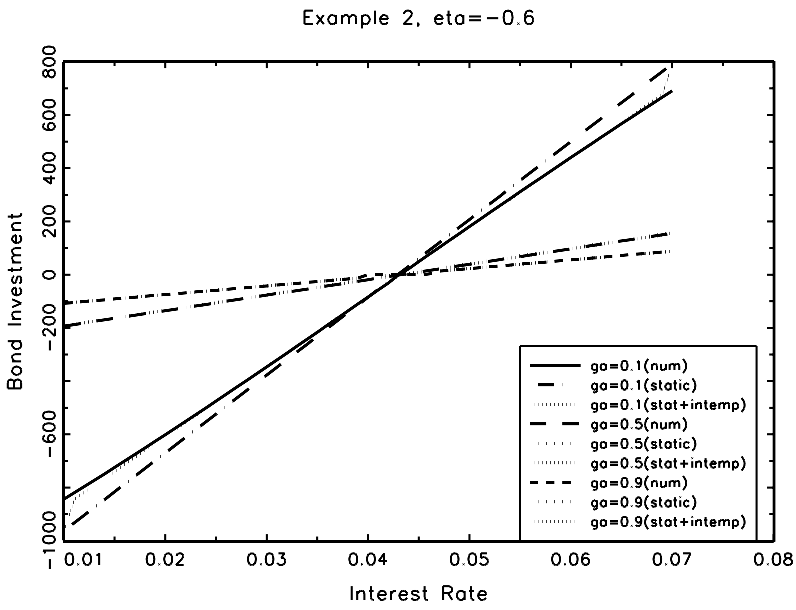

Figure 12: Portfolio Choice for Bond, $\eta = -0.6$

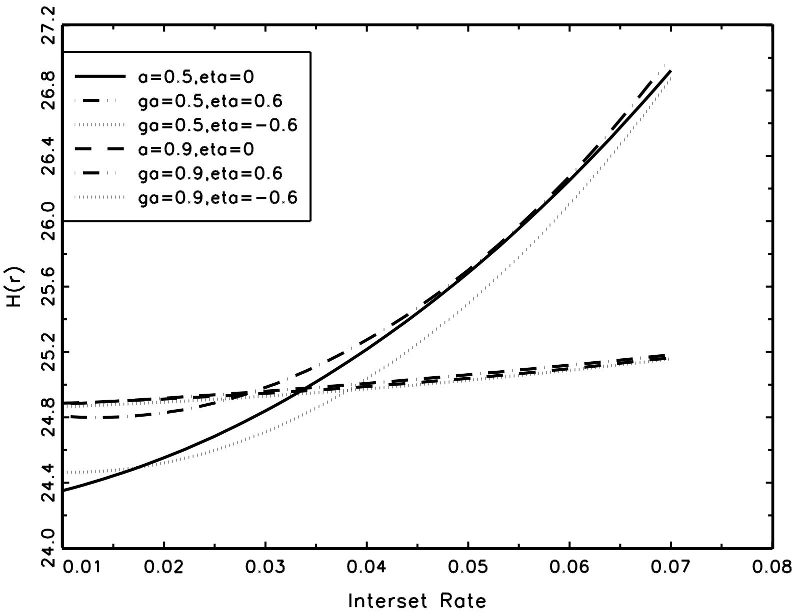

Figure 13: Value Function

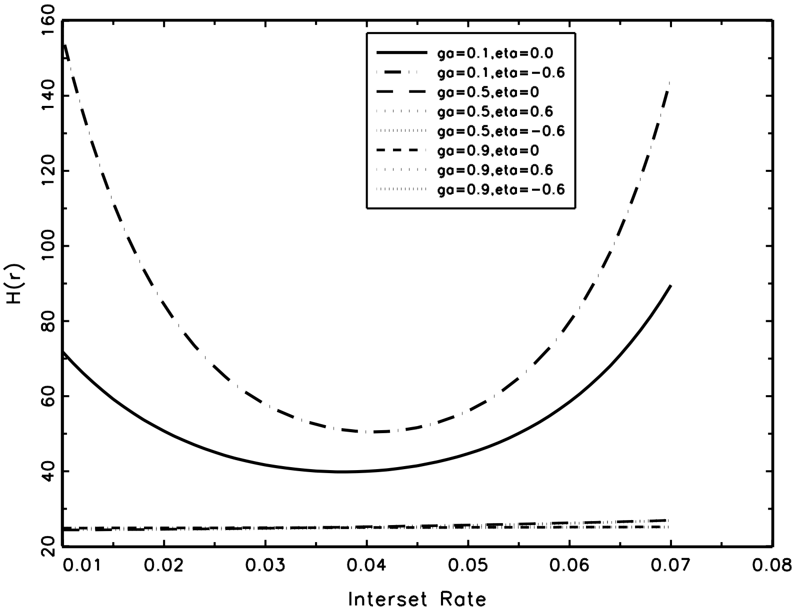

Figure 14: Value Function

Canner, Mankiw and Weil (1997) summarize the type of investment advice given to the general public. They find higher Bond/Stock investment ratios are recommended for more conservative investors. Figures 15 and 16 show this ratio from our numerical exercises. The Bond/Stock ratio is even the largest for $\gamma = 0.1$. This is because the intertemporal effect is positive for $\gamma = 0.1$ and is negative for $\gamma = 0.5$ and $\gamma = 0.9$. It does not seem to support the observations of Canner et al(1997) .

Here we suggest another view to treat this problem. Bonds are preferred by more conservative investors because their payouts are fixed on a desired maturity date. Thus, we should not prescribe a special bond for investment and should allow bonds of all possible maturity dates to be traded. In the one-factor no-arbitrage bond pricing model of Vasicek (1977) every bond of a given maturity date can be generated by the money market account and a fixed given bond. Therefore, concerning the bond market investment we consider the total investment on the bond and on the money market account. Figures 17 − 20 are the Bond/Stock ratios from the viewpoint of the "general bond" portfolio.[30] They correspond with the conventional wisdom: the more conservative investors hold more bonds. In Figures 18 and 20 the bond holdings are negative for large r due to the negative excess return $\mu_s - r$ while the bond holdings are still positive. In this case the general-bond/stock ratios still correspond to the conventional wisdom in absolute value.

The reason for this result is quite simple. In this example, the General-Bond/Stock ratio can be represented by $\frac{1-\alpha_2}{\alpha_2}$. Recall that α_2 is the investment proportion on stock and the sum of all investment proportions is equal to one. The higher general-bond/stock ratio for larger γ is followed by the smaller holding proportion in the stock which can be explained by the risk aversion effect: the factor $\frac{1}{\gamma}$ for the portfolio choice in (58).

[30]We have to separate the interest rate region for two parts because at the interest rate level 0.045 there is no excess return for the stock, therefore there is no stock holding. Due to this the bond/stock ratio behaves unstable in the neighborhood of 0.045.

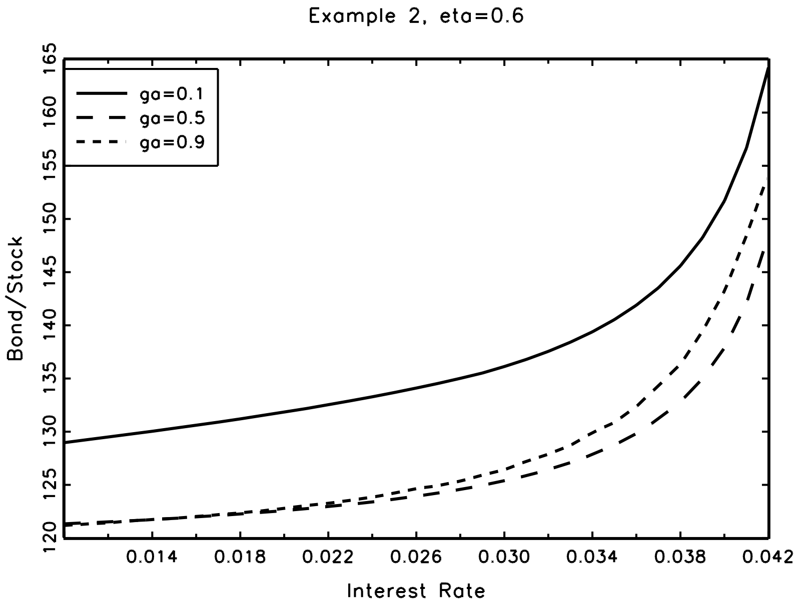

Figure 15: Bond/Stock Ratio, $\eta = 0.6$

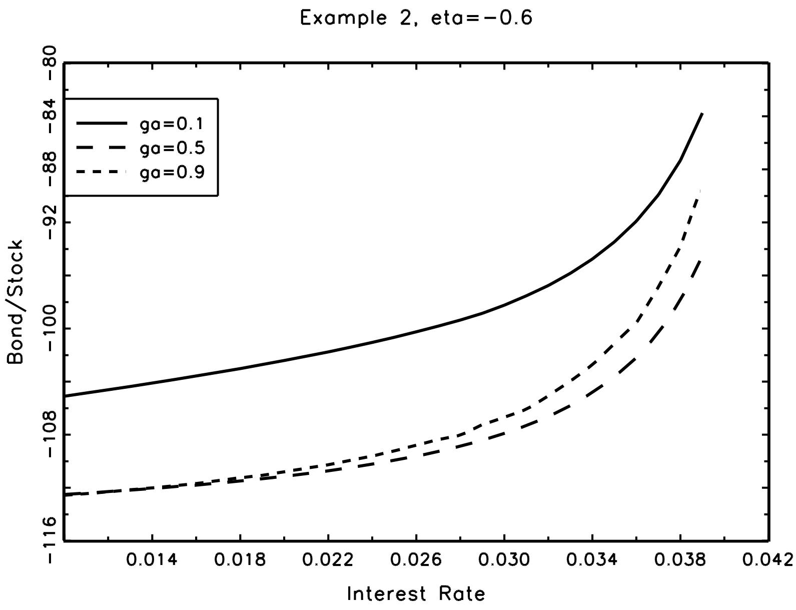

Figure 16: Bond/Stock Ratio, $\eta = -0.6$

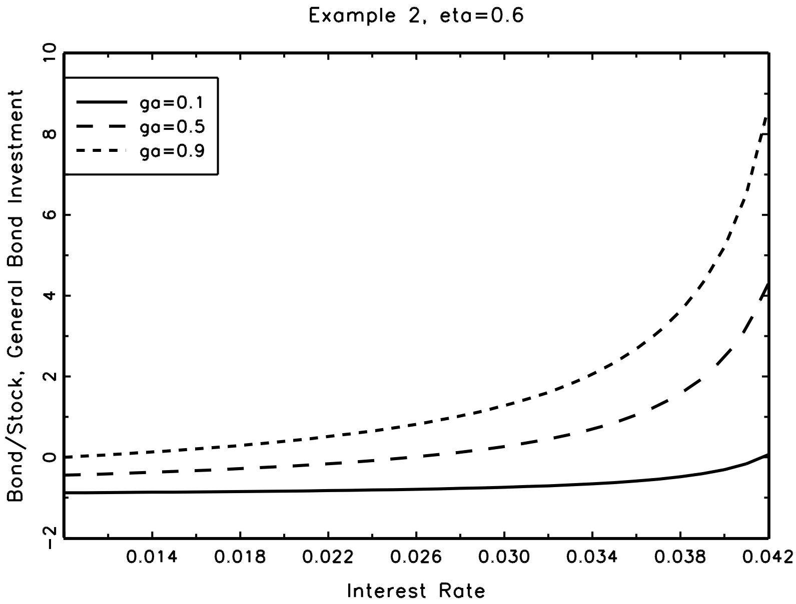

Figure 17: General-Bond/Stock Ratio, $\eta = 0.6$, first region

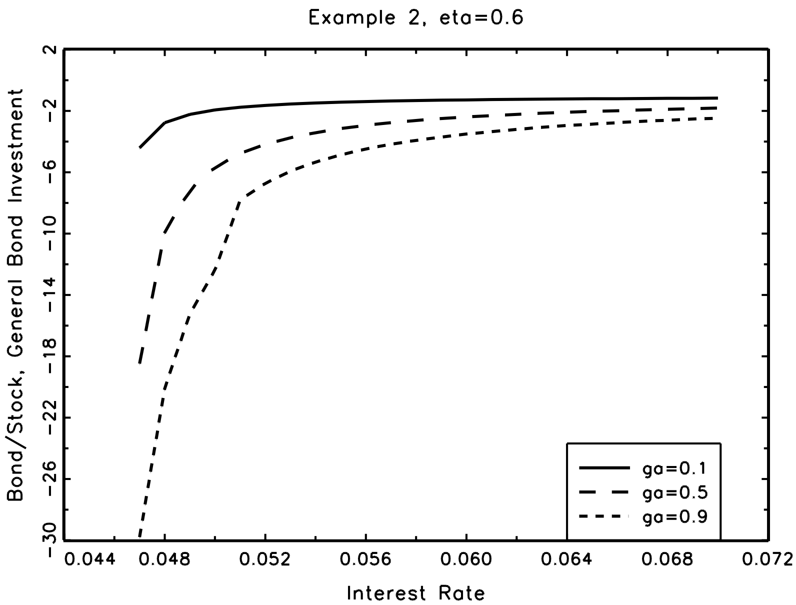

Figure 18: General-Bond/Stock Ratio, $\eta = 0.6$, second region

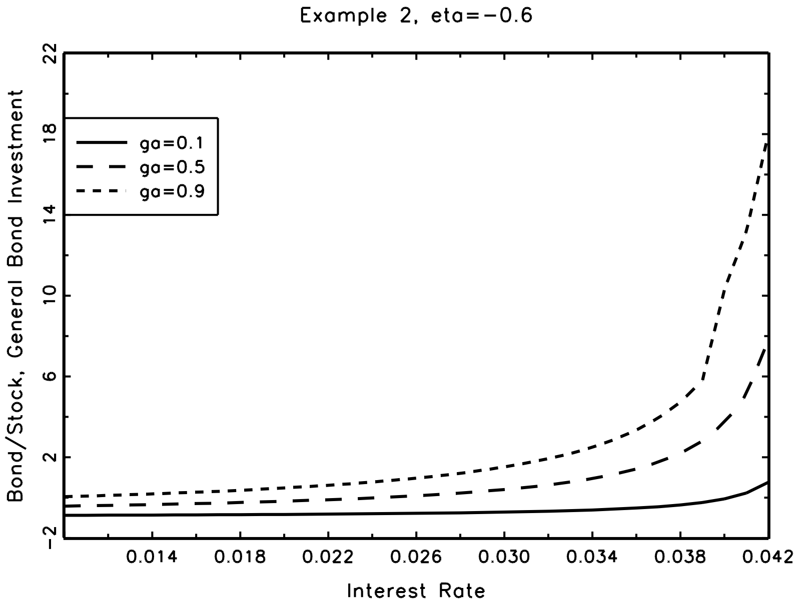

Figure 19: General-Bond/Stock Ratio, $\eta = -0.6$, first region

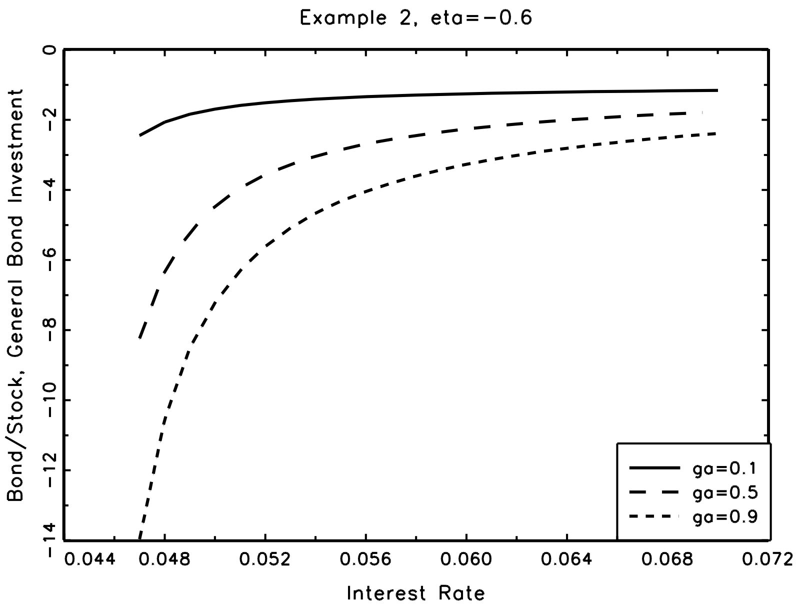

Figure 20: General-Bond/Stock Ratio, $\eta = -0.6$, second region

5 Conclusion

This paper studies the long-term consumption and portfolio decisions in the context of infinite time horizon models. The focus is to investigate the investment allocation of stocks and bonds. The theoretical model is a continuous time model. The instantaneous interest rate is an exogeneuous stochastic process and the bond market obeys the no-arbitrage principle. Using the method of dynamic programming we can separate formally the intertemporal portfolio choice into a static portfolio choice and an intertemporal hedging term additively. The static portfolio choice can be solved easily while the intertemporal term does not have an analytical solution usually. A numerical method – namely, the iteration method – is developed to solve the intertemporal portfolio choice and the intertemporal hedging term. The numerical performance can be checked for some simple cases and it is quite satisfactory.

The results of our numerical exercise show that for most cases, the intertemporal effects are not significant and the static portfolio choices are dominant. The numerical results explain the asset allocation puzzle not quite well. The most conservative investors are willing to hold the largest position of bonds. For future research we will further study whether these two results would hold for more general models. The next step in extending the model would be, for example, a model including inflation, multi-factor interest rates, the stock return with mean-reverting excess return and stochastic volatility.

In this paper we suggest a new way to treat bond investment. In our example using Vasicek's model, we can generate bonds of all possible maturity with the money market account and a given bond. Since we do not have reason to restrict the bond investment to a special bond we should consider the bond investment also to include money. The ratio of the aggregate bond investment to the stock in our numerical exercise behaves the same way as often suggested by financial advisors.

6 Appendix

6.1 Proofs of Section 2

Proof Theorem 1
We write the excess returns of the assets in vector form

$$
\begin{aligned}
(\mu(x,t) - r(x)) &= \begin{pmatrix} \mu_1(x,t) - r(x) \\ \vdots \\ \mu_{d+1}(x) - r(x) \end{pmatrix} \qquad (59) \\
&= \begin{pmatrix} \sum_{j=1}^d \lambda_j(x) B_j(T_1 - t) g_j(x) \\ \vdots \\ \sum_{j=1}^d \lambda_j(x) B_j(T_d - t) g_j(x) \\ \mu_{d+1}(x) - r(x) \end{pmatrix} \\
&= \begin{pmatrix} B_1(T_1 - t) & \cdots & B_d(T_1 - t) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ B_1(T_d - t) & \cdots & B_d(T_d - t) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1(x) g_1(x) \\ \vdots \\ \lambda_d(x) g_d(x) \\ \mu_{d+1}(x) - r(x) \end{pmatrix}.
\end{aligned}
$$

Let

$$
(\tilde{\mu}(x) - r(x)) = \begin{pmatrix} \lambda_1(x) g_1(x) \\ \vdots \\ \lambda_d(x) g_d(x) \\ \mu_{d+1}(x) - r(x). \end{pmatrix}
$$

and

$$
\mathcal{B}_t = \begin{pmatrix} B_1(T_1 - t) & \cdots & B_d(T_1 - t) \\ \vdots & \ddots & \vdots \\ B_1(T_d - t) & \cdots & B_d(T_d - t) \end{pmatrix}. \qquad (60)
$$

We can thus rewrite the equation (59) in

$$
(\mu(x,t) - r(x)) = \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix} (\tilde{\mu}(x) - r(x)) \, . \qquad (61)
$$

The point of this rewriting is to decompose the excess return $\mu(x,t)$ into two parts: the first part contains the information of maturity dates. The second part gives the information of risk premium of each state factor and it is independent of the maturity dates.

Similarly, for the covariance matrix of the asset returns of the vector of the random variables we have the decomposition

$$
\begin{aligned}
& \Omega_t \, dt \\
= \; & \mathbf{E}\left[\begin{pmatrix} -\sum_{i=1}^d B_i(T_1-t) g_i(x) dQ_{it} \\ \vdots \\ -\sum_{i=1}^d B_i(T_d-t) g_i(x) dQ_{it} \\ \sigma_{d+1}(x) dZ_t \end{pmatrix} \begin{pmatrix} -\sum_{i=1}^d B_i(T_1-t) g_i(x) dQ_{it} \\ \vdots \\ -\sum_{i=1}^d B_i(T_d-t) g_i(x) dQ_{it} \\ \sigma_{d+1}(x) dZ_t \end{pmatrix}'\right] \\
= \; & \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix} \tilde{\Omega} \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix}' dt
\end{aligned}
\tag{62}
$$

where $\tilde{\Omega}$ is defined as the covariance matrix

$$
\begin{aligned}
\tilde{\Omega}\, dt = \mathbf{E}&\left[\begin{pmatrix} -g_1(x) dQ_{1t} \\ \vdots \\ -g_d(x) dQ_{dt} \\ \sigma_{d+1}(x) dZ_t \end{pmatrix} \begin{pmatrix} -g_1(x) dQ_{1t} \\ \vdots \\ -g_d(x) dQ_{dt} \\ \sigma_{d+1}(x) dZ_t \end{pmatrix}'\right] \\
= & \begin{pmatrix} g_1^2(x)\nu_{11} & \cdots & g_1(x) g_d(x) \nu_{1d} & -g_1(x)\sigma_{d+1}(x)\eta_1 \\ \vdots & \ddots & \vdots & \vdots \\ g_d(x) g_1(x) \nu_{d1} & \cdots & g_d^2(x)\nu_{dd} & -g_d(x)\sigma_{d+1}(x)\eta_d \\ -\sigma_{d+1}(x) g_1(x) \eta_1 & \cdots & -\sigma_{d+1}(x) g_d(x) \eta_d & \sigma_{d+1}(x)^2 \end{pmatrix} dt
\end{aligned}
$$

which is independent of t.

For the covariance V_{jt} between the noises of the the asset return and the state variable we rewrite it in the similar way. Following the definition of the V_{jt} we obtain

$$
V_{jt} dt = \mathbf{E}\left[\begin{pmatrix} -\sum_{i=1}^d B_i(T_1-t) g_i(x) dQ_{it} \\ \cdots \\ -\sum_{i=1}^d B_i(T_d-t) g_i(x) dQ_{it} \\ \sigma_{d+1}(x) dZ_t \end{pmatrix} g_j(x) dQ_{jt}\right].
$$

Let

$$
\tilde{V}_j dt = \mathbf{E}\left[\begin{pmatrix} -g_1(x) dQ_{1t} \\ \vdots \\ -g_d(x) dQ_{dt} \\ \sigma_{d+1}(x) dZ_t \end{pmatrix} g_j(x) dQ_{jt}\right] = \begin{pmatrix} -g_1(x)\nu_{1j} \\ \vdots \\ -g_d(x)\nu_{dj} \\ \sigma_{d+1}(x)\eta_j \end{pmatrix} g_j(x)\, dt\, .
$$

So the relation between V_{jt} and $\tilde{V}_j$ is given by

$$
V_{jt} = \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix} \tilde{V}_j\, . \tag{63}
$$

Going back to the HJB equation (15) we can see that three terms $(\mu_t - r)'\Omega_t^{-1}(\mu_t -$

$r)$, $(\mu_t - r)'\Omega_t^{-1}V_{jt}$ and $V'_{kt}\Omega_t^{-1}V_{jt}$ are involved in bond returns, hence they are dependent on maturity dates. However, using the the decompositions (61), (62) and (63) we can rewrite these terms in the way, for example,

$$\begin{aligned}
& (\mu_t - r)'\Omega_t^{-1}(\mu_t - r) \\
= \;& (\tilde{\mu} - r)' \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix}' \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix}'^{-1} \tilde{\Omega}^{-1} \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix} (\tilde{\mu} - r) \\
= \;& (\tilde{\mu} - r)'\tilde{\Omega}^{-1}(\tilde{\mu} - r) \,,
\end{aligned}$$

such that they are not dependent on the choice of bonds. Then, the HJB equation (15) can be rewritten to

$$\begin{aligned}
0 \;=\;& e^{-\delta t}U(\psi^* w) + J_t + J_w(r - \psi^*)w + \sum_{i=1}^{m} J_{x_i} f_i \qquad (64) \\
-\;& \frac{1}{2}\frac{J_w^2}{J_{ww}}(\tilde{\mu} - r)'\tilde{\Omega}^{-1}(\tilde{\mu} - r) - \sum_{j=1}^{m} \frac{J_w J_{w,x_j}}{J_{ww}}(\tilde{\mu} - r)'\tilde{\Omega}^{-1}\tilde{V}_j \\
-\;& \frac{1}{2}\sum_{j,k=1}^{m} \frac{J_{w,x_j} J_{w,x_k}}{J_{ww}} \tilde{V}_j'\tilde{\Omega}^{-1}\tilde{V}_k + \frac{1}{2}\sum_{j,k=1} J_{x_j,x_k}\, g_j g_k \nu_{jk} \,,
\end{aligned}$$

which is independent of the chosen bond. Therefore the optimized utility function $J(t, w, x)$ is independent of the choice of bonds.

For the statement (ii) it is straightforward to see that (64) is the HJB equation for the new investment set including the stock (20) and the factor assets (21). Q.E.D.

Proof of Corollary 1.3
Using (61), (62) and (63) to rewrite (12) we obtain

$$\begin{pmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{dt} \\ \alpha_{d+1,t} \end{pmatrix} = -\frac{J_w}{J_{ww}\,W} \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tilde{\Omega}^{-1}(\tilde{\mu} - r) - \begin{pmatrix} \mathcal{B}_t & 0 \\ 0 & 1 \end{pmatrix}^{-1} \sum_{j=1}^{d} \frac{J_{w,x_j}}{J_{ww}\,W}\tilde{\Omega}^{-1} g_j \tilde{V}_j \,. \qquad (65)$$

It can be rewritten further in

$$\begin{pmatrix} \mathcal{B}_t \begin{pmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{dt} \end{pmatrix} \\ \alpha_{d+1,t} \end{pmatrix} = -\frac{J_w}{J_{ww}\,W}\,\tilde{\Omega}^{-1}(\tilde{\mu} - r) - \sum_{j=1}^{d} \frac{J_{w,x_j}}{J_{ww}\,W}\tilde{\Omega}^{-1} g_j \tilde{V}_j \,. \qquad (66)$$

Sine the R.H.S. is independent of the bond choice, then $\mathcal{B}_t\alpha_t$ on the L.H.S is also independent of the bond choice.

For the statement (ii) the wealth dynamics (8) can be decomposed into

$$\frac{dW_t}{W_t} = -\psi_t dt + \frac{dW_t^{(B)}}{W_t} + \frac{dW_t^{(S)}}{W_t}, \qquad (67)$$

where $\frac{dW_t^{(B)}}{W_t}$ represents the change contributed by the fixed income market

$$\begin{aligned}
\frac{dW_t^{(B)}}{W_t} &= r(x)dt + (\alpha_{1t}, \cdots, \alpha_{dt}) \begin{pmatrix} \sum_{j=1} \lambda_j(x) B_j(T_1 - t) g_j(x) \\ \vdots \\ \sum_{j=1} \lambda_j(x) B_j(T_d - t) g_j(x) \end{pmatrix} dt \\
&\quad + (\alpha_{1t}, \cdots, \alpha_{dt}) \begin{pmatrix} -\sum_{j=1} B_j(T_1) g_j(x) dQ_{jt} \\ \vdots \\ -\sum_{j=1} B_j(T_d) g_j(x) dQ_{jt} \end{pmatrix} \qquad (68) \\
&= r(x)dt + \left(\mathcal{B}_t \begin{pmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{dt} \end{pmatrix} \right)' \begin{pmatrix} \lambda_1(x) g_1(x) \\ \vdots \\ \lambda_d(x) g_d(x) \end{pmatrix} dt + \left(\mathcal{B}_t \begin{pmatrix} \alpha_{1t} \\ \vdots \\ \alpha_{dt} \end{pmatrix} \right)' \begin{pmatrix} -g_1(x) dQ_{1t} \\ \vdots \\ -g_d(x) dQ_{dt} \end{pmatrix} \\
&= r(x)dt + (\mathcal{B}_t \alpha_t)' \begin{pmatrix} \lambda_1(x) g_1(x) \\ \vdots \\ \lambda_d(x) g_d(x) \end{pmatrix} dt + (\mathcal{B}_t \alpha_t)' \begin{pmatrix} -g_1(x) dQ_{1t} \\ \vdots \\ -g_d(x) dQ_{dt} \end{pmatrix}. \qquad (69)
\end{aligned}$$

And

$$\frac{dW_t^{(S)}}{W_t} := \alpha_{d+1,t}\big(\mu_{d+1} dt + \sigma_{d+1} dZ_t\big) \qquad (70)$$

represents the change contributed by the stock market.

From (66) we know $\mathcal{B}_t \alpha_t$ and $\alpha_{d+1,t}$ are independent of the bond choice. Then according (69), (70) and Corollary 1.2 the statement (ii) is proved.
Q.D.E.

6.2 Wealth Dynamics

We consider the wealth change in the interval $[t, t+h)$. The timing is that at t the prices $P_i(t)$ are realized. Right after the realization the agents decide the number of shares of i-th asset $N_i(t)$ and the consumption $C(t)$. The The decisions has to satisfy the *self-financing constraint*

$$\sum_{i=0}^{n} N_i(t) P_i(t) + C(t)h = \sum_{i=0}^{n} N_i(t-h) P_i(t).$$

Define the wealth as

$$W(t) = \sum_{i=0}^{n} N_i(t-h) P_i(t).$$

Then

$$\begin{aligned}
W(t+h) - W(t) &= \sum_{i=0}^{n} N_i(t)(\, P_i(t+h) - P_i(t)\,) + \sum_{i=0}^{n} (\, N_i(t) - N_i(t-h)\,) P_i(t) \\
&= (W(t) - C(t)h) \sum_{i=0}^{n} \alpha_i(t) \frac{P_i(t+h) - P_i(t)}{P_i(t)} - C(t)h,
\end{aligned}$$

where

$$\alpha_i(t) = \frac{N_i(t)P_i(t)}{\sum_{j=0}^{n} N_j(t)P_j(t).}$$

Rearrange it we obtain

$$W(t+h) = \big(W(t) - C(t)h\big)(1 + \sum_{i=0}^{n} \alpha_i(t) \frac{P_i(t+h) - P_i(t)}{P_i(t)}).$$

6.3 Separation Theorem for the Agents' Decision

Theorem 3 (Separation Theorem in Agents' Decision) *Let $\Theta_k(\psi, \alpha, x)$ denote a term* [31] *in the iteration (50)*

$$\begin{aligned} & \Theta_k(\psi, \tilde{\alpha}, x) \qquad (71) \\ = \quad & \psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma}\mathbf{E}_t[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)]. \end{aligned}$$

The optimization problem

$$\max_{\psi, \alpha} \Theta_k(\psi, \tilde{\alpha}, x)$$

can solved sequentially. Concretely:

(i) for the case $0 < \gamma < 1$

$$\begin{aligned} \max_{\psi,\alpha} \big\{ \quad & \psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma}\mathbf{E}_t\big[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)\big] \, \big\} \\ = \max_{\psi} \big\{ \quad & \psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma} \\ & \max_{\alpha} \mathbf{E}_t\big[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_2(x, \Delta Q)\big)\big] \, \big\}. \qquad (72) \end{aligned}$$

- *For the case* $\gamma > 1$*,*

$$\begin{aligned} \min_{\psi,\alpha} \big\{ \quad & \psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma}\mathbf{E}_t\big[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)\big] \, \big\} \\ = \min_{\psi} \big\{ \quad & \psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma} \\ & \min_{\alpha} \mathbf{E}_t\big[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_2(x, \Delta Q)\big)\big] \, \big\}. \qquad (73) \end{aligned}$$

Proof We define

$$H_k(\alpha, x) = \mathbf{E}_t\big[\Pi(\alpha, x, \Delta Z, t)^{1-\gamma} S_k\big(\varphi_X(x, \Delta Q)\big)\big].$$

and rewrite

$$\Theta_k(\psi, \alpha, x) = \psi^{1-\gamma}h + e^{-\delta h}(1-\psi h)^{1-\gamma} H_k(\alpha, x).$$

[31] We suppress the argument t here because t is just a constant in the optimization problem.

It is straightforward to see that the first order condition for $\Theta(\psi, \alpha, x)$ with respect to α is identical to the first order condition for $H(\alpha, x)$ with respect to α

$$\frac{\partial \Theta_k}{\partial \alpha} = e^{-\delta h}(1-\psi h)^{1-\gamma}\frac{\partial H_k}{\partial \alpha} = 0.$$

Therefore we can optimize $H(\alpha, x)$ w.r.t α independently.

The remained problem is only to figure out which extreme value of $H_k(\alpha, x)$ will maximize $\Theta_k(\psi, \alpha, x)$.

Let

$$\overline{\Theta}(\psi, H) = \psi^{1-\gamma} h e^{-\delta h}(1-\psi h)^{1-\gamma} H.$$

The necessary and the sufficient condition for the extreme solution ψ is the first order condition [32]

$$\frac{(\psi)^{-\gamma}}{e^{-\delta h}(1-\psi h)^{-\gamma}} = H. \tag{74}$$

So, we can solve ψ in terms of H denote the solution as

$$\overline{\psi}(H) = \frac{(e^{-\delta h}H)^{-1/\gamma}}{1 + h(e^{-\delta h}H)^{-1/\gamma}}. \tag{75}$$

From (75) we observe that $\overline{\psi}(H)$ depends negatively on H.

Evaluating $\overline{\Theta}(\psi, H)$ at the optimal solution $\overline{\psi}(H)$ we obtain

$$\begin{aligned}\overline{\Theta}\big(\overline{\psi}(H), H\big) &= \overline{\psi}(H)^{1-\gamma}h + e^{-\delta h}(1-\overline{\psi}(H)h)^{1-\gamma}H \\ &= \overline{\psi}(H)^{1-\gamma}h + (1-\overline{\psi}(H)h)^{1-\gamma}\frac{\overline{\psi}(H)^{-\gamma}}{(1-\overline{\psi}(H)h)^{-\gamma}} \\ &= \overline{\psi}(H)^{-\gamma}. \end{aligned} \tag{76}$$

Then from (76) we observe that $\overline{\Theta}\big(\overline{\psi}(H), H\big)$ depends positively on H because if H increase, $\overline{\psi}(H)$ decreases, then $\overline{\Theta}\big(\overline{\psi}(H), H\big)$ increases. Therefore, to maximize $\overline{\Theta}(\psi, H)$ we have maximize H. The statement (72) is proved.

For the case $\gamma > 1$ the relationships (75) and (76) still holds. Thus $\overline{\Theta}\big(\overline{\psi}(H), H\big)$ depends still positively on H. The statement (73) is proved.

Q.E.D.

Corollary 3.1 *The notations are as same as in Theorem 3. Let*

$$\begin{aligned} H_k^*(x) &= \max_\alpha H_k(\alpha, x) \quad \textit{for } 0 < \gamma < 1 \\ &= \min_\alpha H_k(\alpha, x) \quad \textit{for } \gamma > 1. \end{aligned}$$

The solution of the optimal $\psi_k(x)$ is given by

$$\psi_k^*(x) = \frac{(e^{-\delta h}H_k(x))^{-1/\gamma}}{1 + h(e^{-\delta h}H_k(x))^{-1/\gamma}} \tag{77}$$

[32]The second derivative w.r.t ψ is negative. This extreme value is a maximum.

and the evolution of the dynamic part of the underlying factors is solved in

$$S_{k+1}(x) = (\psi_k^*(x))^{-\gamma}. \tag{78}$$

References

[1] M.J. Brennan and E.S. Schwartz. A continuous time approach to the pricing of bonds. *Journal of Banking and Finance*, 3:133–155, 1979.

[2] M.J. Brennan, E.S. Schwartz, and R. Lagnado. Strategic asset allocation. *Journal of Economic Dynamics and Control*, 21:1377–1403, 1997.

[3] M.J. Brennan and Y. Xia. Dynamic asset allocation under inflation. *The Journal of Finance*, LVII(3):1201–1238, 2002.

[4] J. Y. Campbell and L. M. Viceira. Who should buy long-term bonds? *The American Economic Review*, 91:99–127, 2001.

[5] J.Y. Campbell, A.W. Lo, and A.C. Mackinlay. *The Econometrics of Financial Markets*. Prinston Press, 1997.

[6] J.Y. Campbell and L. M. Viceira. Consumption and portfolio decisions when expected returns are time varying. *The Quarterly Journal of Economics*, 114:433–495, 1999.

[7] J.Y. Campbell and L.M. Viceira. Strategic asset allocation in continuous-time var. 2001.

[8] J.Y. Campbell and L.M. Viceira. *Strategic Asset Allocation*. Oxford Univery Press, 2002.

[9] N Canner, N.G. Mankiw, and D.N.Weil. An asset allocation puzzle. *The American Ceonomic Review*, 87:181–191, 1997.

[10] C. Chiarella. *An Introduction to Derivative Security Pricing: Techniques, Methods and Applications*. 2004.

[11] J.C. Cox, J.E. Ingersoll, and S.A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, March 1985.

[12] D. Duffie and R. Kan. A yield-factor model of interest rates. *Mathematical Finance*, 6(4):397–406, 1996.

[13] L. Gruene. Numerik optimaler steuerung. Internet, www.math.uni-frankfurt.de/ gruene/teaching/optctrl01.

[14] L. Gruene. An adaptive grid scheme for the discrete hamilton-jacobi-bellman equation. *Numericsche Mathematik*, 75:319–337, 1997.

[15] John Hull and Alan White. Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3(4):573–592, 1990.

[16] John Hull and Alan White. Numerical procedures for implementing term structure models i: Single-factor models. *Journal of Derivatives*, 2:7–16, 1994.

[17] John Hull and Alan White. Numerical procedures for implementing term structure models ii: Two-factor models. *Journal of Derivatives*, 2:37–48, 1994.

[18] M. I. Kamien and N.L. Schwartz. *Dynamic Optimization*. North-Holland, 1991.

[19] T.S. Kim and E. Omberg. Dynamic nonmyopic portfolio behavior. *The Review of Financial Studies*, 9(1):141–161, 1996.

[20] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1994.

[21] J. Liu. Portfolio selection in stochastic environments. 2001.

[22] R. Merton. *Continuous Time Finance*. Blackwell, 1990.

[23] R.C. Merton. Optimum consumption and portfoliio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

[24] R.C. Merton. An intertemporal capital asset pricing model. *Econometrica*, 41(5):867–887, 1973.

[25] C. Munk, C. Sorensen, and T.N. Vinther. Dynamic asset allocation under mean-reverting returns, stochastic interest rates and inflation uncertainty: Are populer recommdendations consistent with rational behavior? *International Review of Economics and Finance*, 13(141-166), 2004.

[26] B. Oksendal. *Stochastic Differential Equations*. Springer, 6 edition, 2000.

[27] A.D. Polyanin and V.F. Zaitsev. *Exact Solutions for Ordinary Differential Equations*. Campman and Hall/CRC, 2 edition, 2002.

[28] O. Vasicek. An equilibrium characterization of the term structure. *Journal of financial economics*, 5:177–188, 1977.