Intertemporal Asset Allocation when the Underlying Factors are unobservable

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Abstract

The aim of this paper is to develop an optimal long-term bond investment strategy which can be applied to real market situations. This paper employs Merton's intertemporal framework to accommodate the features of a stochastic interest rate and the time-varying dynamics of bond returns. The long-term investors encounter a partial information problem where they can only observe the market bond prices but not the driving factors of the variability of the interest rate and the bond return dynamics. With the assumption of Gaussian factor dynamics, we are able to develop an analytical solution for the optimal long-term investment strategies under the case of full information. To apply the best theoretical investment strategy to the real market we need to be aware of the existence of measurement errors representing the gap between theoretical and empirical models. We estimate the model based on data for the German securities market and then the estimation results are employed to develop long-term bond investment strategies. Because of the presence of measurement errors, we provide a simulation study to examine the performance of the best theoretical investment strategy. We find that the measurement errors have a great impact on the optimality of the investment strategies and that under certain circumstance the best theoretical investment strategies may not perform so well in a real market situation. In the simulation study, we also investigate the role of information about the variability of the stochastic interest rate and the bond return dynamics. Our results show that this information can indeed be used to advantage in making sensible long-term investment decisions.

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1 Introduction

The aim of this paper is to construct an optimal long-term strategy for investing in bond securities that would be applicable to trading in real markets. When considering bond investment, there are three reasons why we require an extension of the well-established Capital Asset Pricing Model (CAPM) of Markowitz (1959), Sharpe (1964), and Lintner (1965) for the conduct of bond portfolio management. First, interest rates should be treated as stochastic. One of the main purposes of bond portfolio management is to hedge the risk arising from changing interest rates. Second, the distributions of asset returns should be allowed to vary with time instead of being only identically distributed over time as in the CAPM. It is a well-known fact that the volatility of bond prices decreases as the bonds approach their maturity dates. Third, investors who invest in bond assets are usually more interested in hedging than speculation. They also tend to adopt some long-term investment plan rather than a simple myopic investment strategy. In this paper we will consider the role of the foregoing factors in bond portfolio management within the intertemporal framework of Merton (1971,1973, 1990).

The now extensive literature in intertemporal asset allocation was initiated by Merton (1973) who considered the multi-asset model where the asset returns are driven by some underlying stochastic factors. His essential insight was that investors should consider not only a short period mean-variance trade-off but also a long-term hedging strategy against possible evolutions of the factor dynamics. Thus, the solution of the optimal intertemporal portfolio problem contains two terms, one the regular (mean-variance) term and the intertemporal hedging term.

In order to apply Merton's general framework to the practice of the bond portfolio management, the underlying factors need to be specified. In Kim and Omberg (1996), the factor is a Gaussian risk premium. The three factors in Brennan, Schwartz, and Lagnado (1997) are a short-term interest rate, a long-term interest rate and stock dividends. Brennan and Xia (2002) consider the real interest rate and the expected inflation rate. Brennan, Wang, and Xia (2004), as well as Munk, Sørensen, and Vinther (2004) concentrate on the interest rate and the Sharpe ratio. Due to the introduction of financial derivatives and the increasing complexity of financial trades, stochastic volatility is considered in more recent research, for example, in Liu and Pan (2003).

Unlike the contributions mentioned above, this paper does not specify the underlying factors as specific economic variables a priori but estimates them from observed bond yields. The solution of the optimal bond portfolio problem relies very much on the dynamic setting for the underlying factors. For this reason we let the market data determine the factor dynamics. To this end we employ the dynamic multidimentional term structure model of Duffie and Kan (1996). The Duffie and Kan model is not only analytically tractable but also flexible enough to accommodate empirical features, such as level-dependent volatilities, humped and various other shapes for the yield curve. The essential feature of the Duffie-Kan model from our perspective is the link between the underlying factors and bond yield data. Based on the Duffie-Kan model, we can set up a formula where the underlying factors can be filtered from market bond yield data.

Before we implement the factor estimation, the identification problem needs to be discussed because of the fact that one data generating process may have distinct parameter representations. To solve this problem we need to impose additional conditions on the parameter space so that one data generating process has exactly one parameter representation satisfying the given identification conditions. This paper will give a different parameter representation from the canonical forms of Dai and Singleton (2000). Our representation turns out to have an easier solution to the intertemporal asset allocation problem.

To solve the intertemporal asset allocation problem, Merton (1971) proposed the method of *dynamic programming*. Cox, Ingersoll and Ross (1985) (CIR) give analytical solutions for the square root process. In general, however, there are only a few cases that can be solved analytically. Campbell and Viceira (2002) develop an approximate solution for the log-utility case.[1] Liu (2005) characterizes the conditions on asset returns that support analytical solutions. In this paper, we apply the Feynman-Kac formula to the HJB equation arising from the method of dynamic programming. The solution has an expectation representation, which is similar to the solution obtained by using the static variational method of Cox and Huang (1989).

In order to give an investment recommendation that can be usefully applied to actual market situations, we estimate the bond pricing model based

[1] For the log-utility function, the intertemporal effect vanishes.

on the yield data of the German securities market. Our empirical task is to decide the most appropriate model in the context of the intertemporal asset allocation problem. Within the framework of this paper, the task reduces to the determination of the number of the factors in the Gaussian Duffie-Kan model.

When fitting the model to market data, the theoretical bond pricing formula cannot hold exactly, but only with some measurement errors. This fact has implications for the intertemporal problem since we must take account of the fact that the solutions we have obtained are derived from the model without measurement errors. Therefore, we develop a simulation study to investigate the impact of the measurement errors on the performance of the best theoretical investment strategies. In the simulation study, we also consider investment strategies based on different information about the bond pricing model. We will assume that some agents know the intertemporal feature of the bond prices so that they invest according to the intertemporal strategy, whilst other agents can only observe market bond prices and they adopt an investment strategy based on a risk-return trade-off.

The remainder of the paper is organized as follows. Section 2 sets up the model for the intertemporal asset allocation problem. The first part of Section 2 reviews the Duffie-Kan multifactor term structure model and discusses the model identification problem. The second part develops the optimal intertemporal asset allocation strategy based on the Duffie-Kan model. The form of the solution of the intertemporal problem obtained by using the Feynman-Kac formula is provided. Section 3 presents the empirical study of the bond pricing model where we estimate the Gaussian Duffie-Kan model based on the data for the German securities market. The simulation study is provided in Section 4. The last section draws some conclusions. A number of technical results are gathered in the appendices.

2 The Intertemporal Asset Allocation Problem

In this section we set up the model for the intertemporal decision problem. The intertemporal asset allocation problem is to choose optimal asset allocation strategies in order to maximize agents' long-term expected utility of consumption. The form of the optimal asset allocation strategies depends on the kinds of assets available for investment. We consider an investment opportunity set that only consists of bond assets, and use the Duffie and Kan (1996) framework to model them. The Duffie and Kan model is reviewed in Section 2.1. Section 2.2 reviews the method of dynamic programming proposed by Merton (1971), which we use to solve the intertemporal asset allocation problem, and solves the optimal asset allocation strategies by employing the Feynman-Kac formula.

2.1 Modelling Bond Assets

First, we review the Duffie-Kan affine family briefly and then discuss the identification problem for this family.

2.1.1 The Duffie-Kan affine family

The Duffie-Kan affine family of bond models has the characteristics that the bond price $P(t, \overline{T}, X_t)$ is given by the exponential affine form

$$P(t, \overline{T}, X_t) = e^{-A(\overline{T}-t)-B(\overline{T}-t)^\top X_t} , \tag{1}$$

where t is the current time and $\overline{T}$ is the maturity date. The bond price depends on the current level of the factors X_t. The factors X_t are represented by an n-dimensional stochastic process that will be specified later. All bonds considered in this paper pay no coupons. The coefficients $A(\tau)$ and $B(\tau)^\top = (B_1(\tau), \cdots, B_n(\tau))$ are differentiable functions. The bond payout at the maturity is set to be 1, so that $P(\overline{T}, \overline{T}, X_{\overline{T}}) = 1$. This condition in turn implies that the initial conditions for the coefficients are given by $A(0) = B_i(0) = 0$, for all $i = 1, \cdots, n$.

The *bond yield* $y(t, \overline{T}, X_t)$ is defined as an average return, so it has the affine structure

$$y(t, \overline{T}, X_t) := \frac{\ln P(\overline{T}, \overline{T}, X_{\overline{T}}) - \ln P(t, \overline{T}, X_t)}{\overline{T}-t} = \frac{A(\overline{T}-t)}{\overline{T}-t} + \frac{B(\overline{T}-t)^\top}{\overline{T}-t} X_t . \tag{2}$$

The spot interest rate R_t is defined as the instantaneous yield, which can be represented by

$$R_t = \lim_{s \uparrow t} y(X_s, t-s) = \xi_0 + \xi_1^\top X_t , \tag{3}$$

where we let $\xi_0 = A'(0)$ and $\xi_1 = B'(0)$.

Duffie and Kan (1996) show that the exponential affine bond price (1) is supported by affine dynamics for the underlying factors

$$dX_t = \mathcal{K}(\theta - X_t)dt + \Gamma\sqrt{S_t}dW_t , \tag{4}$$

where θ is an $n \times 1$ constant vector, $\mathcal{K}$ and $\Gamma = (\gamma_{ij})_{i,j=1,\cdots,n}$ are $n \times n$ constant matrices, and S_t is a diagonal $n \times n$ matrix with affine elements $S_i(X_t) = \alpha_i + \beta_i^\top X_t$. The noise term is represented by a *standard* (orthogonal) n-dimensional Wiener process W_t .

2.1.2 A subfamily: The Gaussian factor model

In this paper we only consider a subfamily of the Duffie-Kan family, namely the one where the factor X_t follows an n-dimensional Gaussian process

$$dX_t = \mathcal{K}(\theta - X_t)dt + \Gamma dW_t \ . \tag{5}$$

We require further that the matrix $\mathcal{K}$ be positive definite so that the process X_t is stationary. Also, the matrix $\mathcal{K}$ is required to have distinct eigenvalues. The volatility coefficient matrix Γ is assumed to be full-rank so that any factor noise $\Gamma_i dW_t$, where Γ_i denotes the i-th row in Γ, cannot be substituted by any linear combination of the other factor noises.

Applying Itô's Lemma to the bond price (1), the dynamics of the instantaneous bond return are given by

$$\frac{dP(t, \overline{T}, X_t)}{P(t, \overline{T}, X_t)} = \mu_P(\overline{T} - t, X_t)dt - B(\overline{T} - t)^\top \Gamma dW_t \ , \tag{6}$$

where

$$\mu_P(\tau, X_t) = A'(\tau) + B'(\tau)^\top X_t - B(\tau)^\top \mathcal{K}(\theta - X_t) + \frac{1}{2}\sum_{i,j=1}^{n} B_i(\tau)B_j(\tau)\Gamma_i\Gamma_j^\top \ . \tag{7}$$

The bond market satisfies the standard *no-arbitrage condition*

$$\mu_P(\tau, X_t) - R_t = -B(\tau)^\top \Gamma\lambda \ , \tag{8}$$

for all $\tau > 0$, where λ is an $n \times 1$ constant vector $\lambda = (\lambda_1, \cdots, \lambda_n)$. Each λ_i can be interpreted as the *market price of the factor innovation* W_{it}. The no-arbitrage condition (8) states that the excess return over the riskless return on the left hand side should be equal to the risk premia on the right hand side.

The no-arbitrage condition (8) requires that the coefficients $A(\tau)$ and $B(\tau)$ satisfy the ordinary differential equations

$$
\begin{aligned}
B'(\tau) &= -\mathcal{K}^\top B(\tau) + \xi_1 \,, &(9)\\
A'(\tau) &= (\mathcal{K}\theta - \Gamma\lambda)^\top B(\tau) - \frac{1}{2}\sum_{i,j=1}^{n} B_i(\tau)B_j(\tau)\Gamma_i\Gamma_j^\top + \xi_0 \,. &(10)
\end{aligned}
$$

2.1.3 The model identification problem

When considering a *multifactor term structure model* such as the one given by equations (1), (5), (9) and (10), we are inevitably confronted with an *identification problem*, especially when we do not specify the factors X_t as specific economic variables but rather seek to infer them from market data. The identification problem arises due to the fact that different parameter representations in the multifactor term structure model can generate the same bond prices. This can be illustrated as follows. To a set of factors X_t in the bond pricing formula (1), we can apply a full-rank transformation $\mathcal{L}$ and we still get the same multifactor term structure model based on the transformed factors $X_t^{\mathcal{L}} := \mathcal{L}X_t$ since

$$
y(t, t+\tau, X_t) = \frac{A(\tau)}{\tau} + \frac{B(\tau)^\top}{\tau}X_t = \frac{A(\tau)}{\tau} + \frac{(\mathcal{L}^{-1\top}B(\tau))^\top}{\tau}X_t^{\mathcal{L}} \,.
$$

The transformed factors $X_t^{\mathcal{L}}$ follow the stochastic differential equation

$$
dX_t^{\mathcal{L}} = \mathcal{L}dX_t = \mathcal{L}\mathcal{K}\mathcal{L}^{-1}(\mathcal{L}\theta - X_t^{\mathcal{L}})dt + \mathcal{L}\Gamma dW_t \,,
$$

which is different from the original factor dynamics (5).

To solve the identification problem, we need to impose additional conditions on the parameters $(\theta, \mathcal{K}, \Gamma, \lambda, \xi_0, \xi_1)$ such that for any bond yield expression of the form (2) there exists *only one* parameter representation satisfying those conditions.

Property 1 *Consider the bond yield model (2), where the factors X_t follow the dynamics (5) and the coefficients $B(\tau)$ and $A(\tau)$ satisfy the no-arbitrage conditions (9) and (10). Assume that the parameters $(\theta, \mathcal{K}, \Gamma, \lambda, \xi_0, \xi_1)$ of this model satisfy the identification conditions:*

(i) $\mathcal{K}$ *in (5) is diagonal,*

(ii) θ *in (5) is equal to* $(0, \cdots, 0)^\top$*,*

(iii) ξ_1 *in (9) is equal to* $\big(1, \cdots, 1\big)^\top$*,*

(iv) Γ *in (5) is lower-triangular.*

Then, for each data generating process (2) for $y(t, t+\tau, X_t)$*, there exists only one corresponding parameter representation (up to permutations of the factors* X_t*).*

Our parameter representation given in Property 1 is different from the canonical representation of Dai and Singleton (2000, p.1948) where, in our notation, the matrix $\mathcal{K}$ is lower-triangular while Γ is diagonal. In our representation, each factor X_i has a distinct mean-reverting speed represented by the parameter κ_i while in the Dai and Singleton representation, the factors are stochastic processes independent of each other. These two representations are equivalent in the sense that we can find a full-rank linear matrix to transform one representation to the other and vice verse. The reason why we choose this parameter representation rather than the canonical representation is because of its convenience in solving for the coefficient $B(\tau)$ in equation (9) and the intertemporal optimal strategies that will be introduced later.

The following property solves the coefficients $B_i(\tau)$, $A(\tau)$ in the bond price formula (1) satisfying the identification conditions given in Property 1.

Property 2 *Let* $\kappa_1, \cdots, \kappa_n$ *be the elements on the diagonal of* $\mathcal{K}$*. Then, the coefficients* $B(\tau)$ *and* $A(\tau)$ *satisfying the no-arbitrage conditions (9) and (10) with the parameter restrictions given in Property 1 are solved as*

$$\begin{aligned} B_i(\tau) &= \frac{1}{\kappa_i}(1 - e^{\kappa_i \tau}), \quad \forall i = 1, \cdots, n & (11) \\ \frac{A(\tau)}{\tau} &= \sum_{i=1}^{n} \frac{\Gamma_i \lambda}{\kappa_i}\big(-1 + \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau}\big) + \xi_0 & (12) \\ &\quad -\frac{1}{2}\sum_{i,j=1}^{n} \frac{\Gamma_i \Gamma_j^\top}{\kappa_i \kappa_j}\big(1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau} - \frac{1 - e^{-\kappa_j \tau}}{\kappa_j \tau} + \frac{1 - e^{-(\kappa_i + \kappa_j)\tau}}{(\kappa_i + \kappa_j)\tau}\big), \end{aligned}$$

where $B(\tau) = \big(B_1(\tau), \cdots B_n(\tau)\big)^\top$.

2.2 Optimal Asset Allocation Strategies

2.2.1 The intertemporal model

The intertemporal asset allocation problem considers homogenous agents whose utility is represented by the CRRA (Constant Relative Risk Aversion) utility function

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} \,, \tag{13}$$

where $\gamma > 0$ and $\gamma \neq 1$. Initially the agents have endowment V_0 where V_t represents wealth at time t. The objective of these agents is to maximize the *expected future utility*

$$\mathbf{E}_0\left[e^{-\delta T} U(V_T)\right] . \tag{14}$$

The agents maximize their objective (14) by choosing an investment plan $(\alpha_{1t}, \cdots, \alpha_{nt})$ for each moment $t \in [0, T]$. Each α_{it} represents the investment proportion in the i-th bond relative to the total wealth level.

For the multifactor no-arbitrage bond model introduced above, we can choose n bonds with distinct maturity dates $\overline{T}_1, \cdots, \overline{T}_n$, as many as the number of the factors, to span all bond returns, see, for example, Chiarella (2004). In other words, any bond can be replicated by a portfolio consisting of these n chosen bonds.

Let $P_{it} := P(t, \overline{T}_i, X_t)$ denote the price of the i-th bond maturing at time $\overline{T}_i$. From (6), the bond return dynamics can be represented in the vector form

$$\begin{pmatrix} \frac{dP_{1t}}{dP_{1t}} \\ \vdots \\ \frac{dP_{nt}}{P_{nt}} \end{pmatrix} = \mu_t dt + \Sigma_t dW_t \,,$$

where

$$\mu_t := \begin{pmatrix} \mu_P(\overline{T}_1 - t, X_t) \\ \vdots \\ \mu_P(\overline{T}_n - t, X_t) \end{pmatrix} \tag{15}$$

$$\Sigma_t := -\mathcal{B}_t \Gamma \,, \text{with } \mathcal{B}_t := \begin{pmatrix} B_1(\overline{T}_1 - t) & \cdots & B_n(\overline{T}_1 - t) \\ \vdots & \ddots & \vdots \\ B_1(\overline{T}_n - t) & \cdots & B_n(\overline{T}_n - t) \end{pmatrix} . \tag{16}$$

Then, the evolution of wealth due to the the investment plan $(\alpha_{0t}, \cdots, \alpha_{nt})$ can be represented by[2]

$$\frac{dV_t}{V_t} = R_t dt + \alpha_t^\top \Big((\mu_t - R_t \underline{\mathbf{1}}) dt - \mathcal{B}_t \Gamma dW_t \Big) , \tag{17}$$

where $\alpha_t^\top := (\alpha_{1t}, \cdots, \alpha_{nt})$, and $\underline{\mathbf{1}} := (1, \cdots, 1)^\top$.

The remaining investment proportion $\alpha_{0t} := 1 - \sum_{i=1}^n \alpha_{it}$, which can be positive or negative, represents the investment position in the money market which earns the riskless return R_t.

2.2.2 The solution via dynamic programming

Let $J(t, T, V_t, X_t)$ be the value function, which is defined as the maximized objective function (14) for the sub-period $[t, T]$, so that

$$J(t, T, V_t, X_t) = \max_{\alpha_s; t \le s \le T} \Big\{ \mathbf{E}_t \Big[e^{-\delta T} U(V_T) \Big] \Big\} . \tag{18}$$

The value function depends on the wealth V_t and the level of the underlying factor X_t at the initial time of the sub-period. The *Hamilton-Jacobi-Bellman* (HJB) equation[3] characterizes the first order condition that yields the optimal decisions and is given by

$$\begin{aligned} 0 = \max_{\alpha_t} \quad \Big\{ & \frac{\partial}{\partial t} J + \big(R_t + \alpha_t^\top (\mu_t - R_t \underline{\mathbf{1}}) \big) J_V V_t \\ & + \frac{1}{2} \alpha_t^\top \Sigma_t \Sigma_t^\top \alpha_t J_{VV} V_t^2 + (\theta - X_t)^\top \mathcal{K}^\top J_X \\ & + \alpha_t^\top \Sigma_t \Gamma^\top J_{VX} V_t + \frac{1}{2} \sum_{i,j=1}^n \Gamma_i \Gamma_j^\top J_{X_i, X_j} \Big\} . \end{aligned} \tag{19}$$

We observe that during the time period $s \in [t, T]$ the factor dynamics for X_s, given in (5), are independent of the wealth level V_s. Furthermore, the percentage wealth change dV_s / V_s, defined by (17), is also independent of the wealth level V_s. Given the just-stated independence of V_t, it turns out that the initial wealth level V_t does not affect the optimal decisions α_s^*

[2]See Merton (1971) .

[3]The HJB equation states that the optimal lifetime utility over $[t, T]$ should be equal to the optimal momentum utility for a short time interval $[t, t + dt)$ plus the optimal lifetime utility over $[t + dt, T]$. See Kamien and Schwartz (1991: 264-271) for a heuristic discussion and Chapter 11 in Øksendal(2000) for a rigorous derivation.

over the period $s \in [t, T]$, and it can be treated as a scalar multiplier of the intertemporal optimization problem (18). So, we can rewrite the value function as

$$\begin{aligned}
& J(t, T, V_t, X_t) \\
= & \max_{\alpha_s; t \le s \le T} \left\{ V_t^{1-\gamma} e^{-\delta T} \mathbf{E}_t[U(\frac{V_T}{V_t})] \right\} \\
= & V_t^{1-\gamma} J(t, T, 1, X_t) = e^{-\delta t} U(V_t) \Phi(t, T, X_t)^\gamma \;, \qquad (20)
\end{aligned}$$

where we set

$$\Phi(t, T, X_t)^\gamma := e^{\delta t}(1-\gamma) J(t, T, X_t, 1) \;.$$

The product form (20) separates V_t from the other dependent variables.

Using the product form (20), the HJB equation (19) may be written as[4]

$$\begin{aligned}
0 = \max_{\alpha_t} \quad \Bigg\{ \quad & -\delta J + (1-\gamma)\big(R_t + \alpha_t^\top(\mu_t - R_t \underline{\mathbf{1}})\big) J \\
& -\frac{1}{2}(1-\gamma)(-\gamma)\alpha_t^\top \Sigma_t \Sigma_t^\top \alpha_t J + \gamma(\theta - X_t)^\top \mathcal{K}^\top \frac{\Phi_{X_i}}{\Phi} J \\
& +(1-\gamma)\gamma \alpha_t^\top \Sigma_t \Gamma^\top \frac{\Phi_{X_i}}{\Phi} J \\
& +\frac{1}{2}\sum_{i,j=1}^{n} \Gamma_i \Gamma_j^\top \Big(\gamma(\gamma-1)\frac{\Phi_{X_i}}{\Phi}\frac{\Phi_{X_j}}{\Phi} + \gamma \frac{\Phi_{X_i X_j}}{\Phi}\Big) J \Bigg\} \;,
\end{aligned}$$

from which the first order condition (FOC) for α_t leads to the expression for the optimal α_t given by

$$\alpha_t^* = \underbrace{\frac{1}{\gamma}(\Sigma_t \Sigma_t^\top)^{-1}(\mu_t - R_t \underline{\mathbf{1}})}_{\text{Mean-Variance Efficient}} + \underbrace{(\Sigma_t \Sigma_t^\top)^{-1} \Sigma_t \Gamma^\top \frac{\Phi_X}{\Phi}}_{\text{Intertemporal Hedging}} \;. \qquad (21)$$

The first term in the solution of the optimal portfolio (21) is the *mean-variance* efficient portfolio, where the agents' portfolio decisions are based

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[4] We abbreviate $J(t, T, X_t)$ to J, and use the following equalities

$$\begin{aligned}
\frac{\partial}{\partial t} J_t &= -\delta J + \gamma(\frac{\partial}{\partial t}\Phi)\frac{1}{\Phi} J \;, & J_X &= \gamma \frac{\Phi_X}{\Phi} J \;, \\
J_V V &= (1-\gamma) J \;, & J_{VV} V^2 &= (1-\gamma)(-\gamma) J \;, \\
J_{VX} V &= (1-\gamma)\gamma \frac{\Phi_{X_i}}{\Phi} J \;, & J_{X_i, X_j} &= \Big(\gamma(\gamma-1)\frac{\Phi_{X_i}}{\Phi}\frac{\Phi_{X_j}}{\Phi} + \gamma \frac{\Phi_{X_i X_j}}{\Phi}\Big) J \;.
\end{aligned}$$

on the return-risk trade-off. The second term arises only in an intertemporal framework and is due to the variation of the factors represented by the volatility coefficient Γ in the factor dynamics (5). In a static framework, Γ is equal to zero, then this term vanishes. This decomposition of the optimal intertemporal portfolio decision is one of the profound contributions of Merton (1971,1973).

Applying the equation (21) to the HJB equation (19), the HJB equation can be transformed into the partial differential equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Phi + \Big(\mathcal{K}(\theta - X_t) + \frac{1-\gamma}{\gamma}\Gamma\Sigma_t^{-1}(\mu_t - R_t\underline{\mathbf{1}})\Big)^\top \Phi_X \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{X_iX_j}\Gamma_i\Gamma_j^\top \\ &\quad + \Phi\Big(-\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}R_t + \frac{1-\gamma}{2\gamma^2}(\mu_t - R_t\underline{\mathbf{1}})^\top(\Sigma_t\Sigma_t^\top)^{-1}(\mu_t - R_t\underline{\mathbf{1}})\Big)\,. \end{aligned} \tag{22}$$

For the finite time case we let the final utility function be equal to the temporary utility function

$$J(T,T,V_T,X_T) = U(V_T)\,.$$

Thus, the boundary condition for the multiplicative part $\Phi(t,T,X_t)$ is

$$\Phi(T,T,X_T) = 1\,, \tag{23}$$

due to the definition of Φ in equation (20).

So, the problem of intertemporal asset allocation now reduces to the problem of solving the nonlinear second order partial differential equation (22) for Φ.

2.2.3 Solving the HJB equation via the Feymann-Kac formula

To simplify the HJB equation (22), we let

$$h_t := -\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}R_t + \frac{1-\gamma}{2\gamma^2}\lambda^\top\lambda\,. \tag{24}$$

Together with the fact that

$$\Sigma_t^{-1}(\mu_t - R_t) = \lambda\,,$$

which is based on the no-arbitrage condition (8), the HJB equation can be rewritten further as

$$0 = \frac{\partial}{\partial t}\Phi + \Big(\mathcal{K}(\theta - X_t) + \Gamma\frac{1-\gamma}{\gamma}\lambda\Big)^{\top}\Phi_X + \frac{1}{2}\sum_{i,j=1}^{n}\Phi_{X_iX_j}\Gamma_i\Gamma_j^{\top} + \Phi h_t \,. \quad (25)$$

The application of the the Feymann-Kac formula[5] to represent the solution is straightforward and involves associating the HJB equation (25) with the partial differential equation (59) in the Appendix.

We provide the solution here and the proof is given in Section 6.1 of the appendix.

Property 3 *The solution $\Phi(t, T, X_t)$ for the partial differential equation (25) with the boundary condition (23) is given by the expectation operator representation*

$$\Phi(t, T, X_t) \quad = \quad \mathbf{E}_{t,X_t}\Big[e^{\int_t^T h(X_s,s)ds}\frac{d\hat{\mathcal{P}}_T}{d\mathcal{P}_T}\Big] \,, \quad (26)$$

where the Radom-Nikodym derivative appearing in (26) is given by

$$\frac{d\hat{\mathcal{P}}_T}{d\mathcal{P}_T} = \exp\Big(\frac{1-\gamma}{\gamma}\lambda^{\top}(W_T - W_t) - \frac{(1-\gamma)^2}{2\gamma^2}\lambda^{\top}\lambda(T-t)\Big) \,, \quad (27)$$

and $\mathbf{E}_{t,X_t}$ is the expectation operator with respect to the process $X_s, t \leq s \leq T$, satisfying (5) with initial value X_t.

The reader may note that the expectation operator expression (26) is very similar to the martingale solution obtained by the static variational method of Cox and Huang (1989). It can be shown that the expectation operator representation (26) is equivalent to the solution of Cox and Huang (1989) in Hsiao (2006). Here we provide another way to obtain the martingale solution.

3 Estimating the Factors based on Market Data

In this section we estimate the parameters and the unobservable stochastic factors in the bond pricing formula (2) based on actual market data. The coefficients $A(\tau)$ and $B(\tau)$ in (2) are given in equations (12) and (11) and

[5]See Theorem 1 in Section 6.2 of the Appendix.

the factor X_t follows the dynamics (5). We will determine parameter values $(\mathcal{K}, \Gamma, \lambda, \xi_0)$ in order to fit the observed bond yield data $y(t, \overline{T}, X_t)$. The parameter values are determined by the maximum likelihood method.

The bond yield data are obtained from the Homepage of Deutsche Bundesbank (German Federal Bank)[6]. The yield data are derived from the interest rates on Federal securities using the method of Nelson and Siegel[7]. The bond yields used here are medium- and long-term ones with time to maturity of 1 year, 3, 5, 8, and 10 years[8] corresponding to our purpose for constructing long-term asset allocation strategies. We have chosen the time horizon January 03, 2003 to February 10, 2005 because the bonds were actively traded in this period. All data are available on a daily basis, so there are 535 observation points. Figure 1 plots the data and their descriptive statistics are given in Table 1.

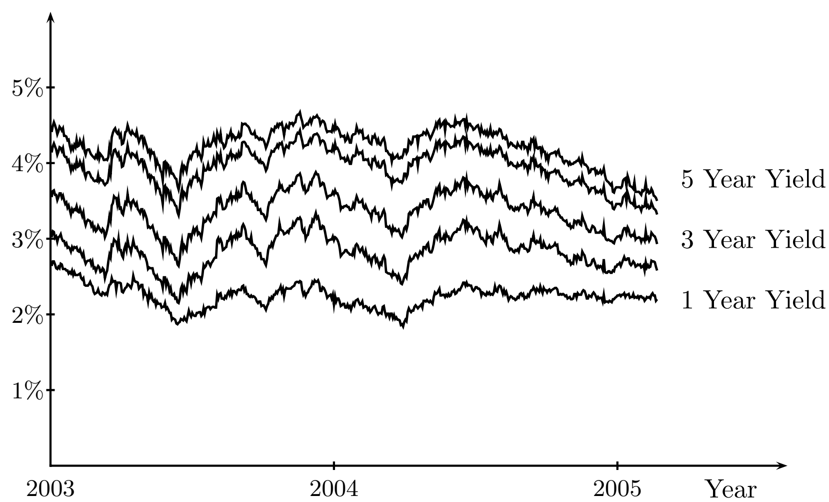


Figure 1: German Yield Curve

[6]The data source is located at "http://www.bundesbank.de/statistik/statistik.en.php", then click "Time Series Database", then "Interest Rates", then "capital market", then "Term structure of interest rates in the debt securities market - estimated values", then "Yields, derived from the term structure of interest rates, on listed Federal securities with annual coupon payments (monthly and daily data)".

[7]See the Monthly Report of the Deutsche Bundesbank, October 1997: 61-66.

[8]Those time series that are labelled by wt3211, wt3215, wt3219, wt3225, and wt3229 respectively.

bond yields	1Y	3Y	5Y	8Y	10Y
mean	0.0224	0.0282	0.0336	0.0395	0.0422
st. deviation	0.00151	0.00224	0.00249	0.00254	0.00256

Table 1: Descriptive Statistics of Bond Yields

In order to estimate the unobservable common factors in the observed bond yield data, we employ the Kalman filter[9]. The *observation equation* of the Kalman filter is the bond yield formula (2) to which measurement errors $\epsilon_t^{\tau_i}$ are added, so that

$$y(t, t+\tau_i, X_t) = \frac{A(\tau_i)}{\tau_i} + \frac{B(\tau_i)^\top}{\tau_i} X_t + \epsilon_t^{\tau_i} \ , \tag{28}$$

where $\{\tau_i, i = 1, \cdots, 5\} = \{1, 3, 5, 8, 10 \text{ years}\}$ correspond to the observed data given above. Reasons for the existence of measurement errors might include the unobservability of X_t, real market frictions, or imperfections in the model itself. The measurement errors $\epsilon_t^{\tau_i}$ are assumed to be identical, independently, and $\mathcal{N}(0, \sigma_\epsilon)$-distributed for all observation time points t and all time to maturity τ_i. Also, they are assumed to be independently distributed with respect to the factor X_t.

The *state equation* of the Kalman filter is given by the factor dynamics (5) where the parameters have to satisfy the normalization conditions in Property 1. Because the market data are discrete-time while the factor dynamics follow the continuous-time dynamics (5), we need to solve the stochastic differential equation (5) between observation times t and $t + \Delta$. Thus, we represent the factor dynamics by the solution[10]

$$X_{t+\Delta} = e^{-\mathcal{K}\Delta} X_t + \int_0^\Delta e^{-\mathcal{K}(\Delta - u)} \theta du + \int_0^\Delta e^{-\mathcal{K}(\Delta - u)} \Gamma_i dW_{t+u} \ . \tag{29}$$

We have developed estimation programs based on the software package "TSM"(Time Series Modelling), which is implemented in the programming language "GAUSS"[11]. For the model estimation we set one time unit equal to one year, so that the time step for daily data is about 1/250.

[9] See Section 6.2 of the Appendix.
[10] See Kloeden and Platen(1992).
[11] For information about "GAUSS" and "TSM" see http://www.aptech.com.

In order to determine how many common factors X_t should be chosen for the underlying dynamics for the bond yields, we implement the model estimation for one-, two-, and three-dimensional factors X_t and then choose the best model according to information criteria, namely, the Akaike, Bayesian and Hannan-Quinn information criteria.

At a maximum, the gradient of the log-likelihood is equal to zero. In the numerical implementation, the convergence tolerance for the gradient is set to be 10^{-5}. However, with this setting, we were not able to obtain convergence for the three-dimensional factor model so we relaxed the convergence tolerance to 0.07 in order to obtain a set of parameter estimates.

All estimation results are listed in Table 2. The Kalman filter estimates of the one-, two-, and three-factor models are plotted in Figures 2, 3 and 4 respectively.

	1-factor Estimates (T-stat.)	2-factor Estimates (T-stat.)	3-factor Estimates (T-stat.)
κ_1	0.0974 (22.18)	0.1431 (10.87)	0.0499 (218.34)
κ_2		0.7748 (18.70)	0.5587 (216.00)
κ_3			0.5991 (328.73)
Γ_{11}	0.0064 (27.10)	0.0110 (16.63)	0.0177 (2554.51)
Γ_{21}		-0.0105 (-6.59)	-0.4583 (-446.13)
Γ_{22}		0.0035 (36.61)	0.5087 (395.88)
Γ_{31}			0.4447 (238.76)
Γ_{32}			-0.5153 (-373.10)
Γ_{33}			0.0072 (397.43)
λ_1	-0.9362 (-13.55)	-0.7561 (-0.95)	-0.1319 (-3.97)
λ_2		-3.7973 (-5.79)	-0.5867 (-1.66)
λ_3			-0.7291 (-1.05)
ξ_0	0.0231 (37.03)	0.0155 (0.33)	0.0040 (3.42)
σ_ϵ	0.0013 (71.07)	0.0006 (41.06)	0.0001 (18.31)

Table 2: Estimated Parameters and T-statistics

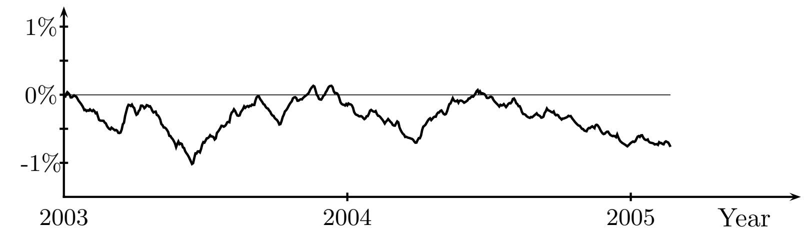


Figure 2: The filtered factors in the one-factor model

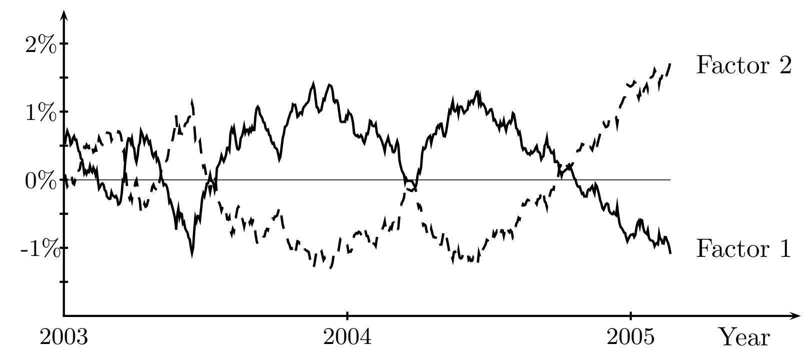


Figure 3: The filtered factors in the two-factor model

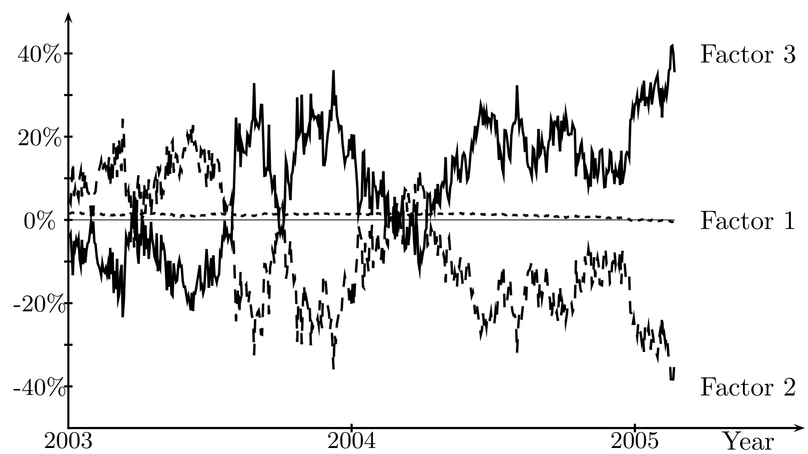


Figure 4: The filtered factors in the three-factor Model

With regard to the estimates we make a number of observations. First, all mean-reverting parameters κ_i are quite small for all three models. Second, the two innovations in the two-factor model, which are represented by the noise terms $\Gamma_{11}W_{1t}$ and $\Gamma_{21}W_{1t} + \Gamma_{22}W_{2t}$, are highly (negatively) correlated with the correlation coefficient -0.9485 based on the estimation values in Table 2. This high correlation of the factor innovations leads to high correlation of the factors processes X_{1t}, X_{2t}, as shown in Figure 3. Third, a similar high correlation can be also found between the second and the third factor innovations in the three-factor model, which are represented by the noise terms $\sum_{i=1}^{2} \Gamma_{2i}W_{it}$ and $\sum_{i=1}^{3} \Gamma_{3i}W_{it}$. The correlation coefficient is equal to -0.9997. The estimated factors are displayed in Figure 4. In the three-factor estimation we also observe that the estimated second and third factors vary over an abnormally large scale compared with the observed market yields.

In Table 3, the statistical performance of the three different models is compared. The "rel. fitting errors" denote the squared quadratic sum of the fitting errors relative to the bond yields' standard deviation, for example, the fitting error of the bond yield with one year maturity in the one-factor model amounts to 0.6281% relative to the standard deviation. The second part of the table provides the results of the three *information criteria*. The value of an information criterion expresses an adjusted goodness of fit with the penalty for utilization of the degrees of freedom, see for example Burnham and Anderson (2004). The results of Table 3 show that the three-factor model is the best statistical model among the three. It has significantly the smallest fitting errors and the smallest values for all information criteria.

rel. fitting errors	$\tau = 1$Y	$\tau = 3$Y	$\tau = 5$Y	$\tau = 8$Y	$\tau =$10Y
1-Factor	0.6281%	0.0652%	0.0725%	0.0813%	0.1441%
2-Factor	0.0228%	0.0715%	0.0181%	0.0122%	0.0277%
3-Factor	2.03×10^{-6}	4.56×10^{-6}	3.67×10^{-6}	1.32×10^{-6}	1.45×10^{-6}

Information criteria	Akaike (AIC)	Bayesian (BIC)	Hannan-Quinn
1-Factor	-10.38	-10.36	-10.37
2-Factor	-11.71	-11.69	-11.70
3-Factor	-13.89	-13.86	-13.88

Table 3: Comparison of the performance of the various models

Overall, the empirical investigation of the bond yield model (2) has given a diversity of results. On the one hand, the graphs in Figure 3 seem to suggest that the second factor is redundant because the trajectories of the two factors are almost like a mirror image of each other. The estimated factor trajectories in the three-factor model fluctuate on a wide scale that is much larger then that of the bond yields themselves. On the other hand, from the statistical point of view, however, it seems that the more factors, the better the statistical performance among the three models.

The estimated models will be used in Section 4 for the simulation study of portfolio performance. We discard the the three-factor model because of its wild behavior, which might lead to extreme investment strategies. We employ the estimation results of the one- and two-factor models.

4 Optimal Portfolios and Simulation Study

In this section we give explicit forms for the optimal intertemporal portfolio strategies. We then undertake a simulation study of portfolio performance based on the estimation results of Section 3.

Given the solution of the value function in Section 2.2.3, we give explict forms of the optimal intertemporal portfolio strategies in Section 4.1. Those optimal strategies, however, are constructed without measurement errors in the pricing formula (28). In order to apply these optimal strategies to real market situations, we need to take account of the existence of the measurement errors in the pricing formulas. Then, the following questions arises naturally. Do the best theoretical (intertemporal) investment strategy still perform well in the presence of the measurement errors? How do the measurement errors affect performance of the strategies? In Section 4.2 we will provide a simulation study that seeks to answer these questions. In the simulation study we are also interested in the problem of partial information, when investors do not have an intertemporal model for investment planning but rather follow the conventional strategy of choosing the mean-variance efficient (MVE) portfolios. We will compare the MVE strategies with the intertemporal optimal strategies.

4.1 Optimal Portfolio without Measurement Errors

Property 4 *The explicit solution of* $\Phi(t,T,x)$ *satisfying the expectation operator representation (26) where the parameters* $\left(\mathcal{K},\Gamma,\lambda,\xi_0\right)$ *satisfy the iden-*

tification conditions given in Property 1, is given by

$$\Phi(t,T,X_t) \;=\; \tilde{f}(t,T)e^{\frac{1-\gamma}{\gamma}B(T-t)X_t}\,, \tag{30}$$

where

$$\begin{aligned}\ln \tilde{f}(t,T) \;=\;& -\frac{\delta}{\gamma}(T-t)+\frac{1-\gamma}{2\gamma^2}\lambda^\top\lambda(T-t)+\frac{1-\gamma}{\gamma}\xi_0(T-t)\\ &+(\frac{1-\gamma}{\gamma})^2\int_t^T B(T-s)^\top\Sigma\lambda ds\\ &+\frac{1}{2}(\frac{1-\gamma}{\gamma})^2\int_t^T B(T-s)^\top\Sigma\Sigma^\top B(T-s)ds\,,\end{aligned}$$

and $B(\tau)^\top = \big(\frac{1}{\kappa_1}(1-e^{-\kappa_1\tau}),\cdots,\frac{1}{\kappa_n}(1-e^{-\kappa_n\tau})\big)$.

Using the result of Property 4, we can obtain the analytical solution for the optimal portfolio given in equation (21) in our case of the optimal bond portfolios, where the derivative in the intertemporal hedging term in the formula (21) is now given by

$$\frac{\Phi_X(t,T,X_t)}{\Phi(t,T,X_t)} = \frac{\partial}{\partial X}\ln\Phi(t,T,X_t) = \frac{1-\gamma}{\gamma}B(T-t)\,. \tag{31}$$

Substituting (31) into the solution α_t^* of the optimal portfolio given in (21), we obtain the analytical representation for the optimal investment strategy in the form

$$\alpha_t^* = \underbrace{\frac{1}{\gamma}(\Sigma_t^\top)^{-1}\lambda}_{\text{Mean-Variance Efficient}} + \underbrace{\frac{1-\gamma}{\gamma}(\Sigma_t^\top)^{-1}\Gamma^\top B(T-t)}_{\text{Intertemporal Hedging}}\,, \tag{32}$$

where we recall that Σ_t has been defined in (16). We remark here that due to the log-linear form of the factor X_t in the solution of the value function (30), the intertemporal hedging term turns out not to depend on the level of the factors. The mathematical reason for this is that the factor follows a mean-reverting Gaussian process and so depends linearly on its past, as shown in the solution (29).

Although the intertemporal hedging term does not directly depend on the factor level, it is still affected by the intertemporal behavior of the factors represented by the mean-reverting parameters $\mathcal{K}$ and the variation Γ. The intertemporal hedging effect is more significant,

(i) when the investors are more risk-averse (large γ),

(ii) when the investment horizon is long (large $T-t$),

(iii) when the factor is more like a random walk process (small mean reversion speed κ_i), and

(iv) when the mean-variance portfolio is not too dominant compared to the intertemporal hedging term (mathematically, we need to compare the scale of the market price of risk $|\lambda|$ with the volatility of the long term bond $B(T-t)^\top\Gamma$).

Furthermore, the optimal wealth based on the optimal portfolio evolves according to

$$\begin{aligned}\frac{dV_t^*}{V_t^*} &= R_t dt + \alpha_t^{*\top}\Big((\mu - R_t\underline{\mathbf{1}})dt + \Sigma_t dW_t\Big) \qquad (33)\\ &= R_t dt + \left(\Big(\frac{1}{\gamma}\lambda^\top + \frac{1-\gamma}{\gamma}B(T-t)^\top\Gamma\Big)\Sigma_t^{-1}\right)\left(\Sigma_t\big(\lambda dt + dW_t\big)\right)\\ &= R_t dt + \frac{1}{\gamma}\big(\lambda^\top\lambda dt + \lambda^\top dW_t\big) + \frac{1-\gamma}{\gamma}B(T-t)^\top\Gamma(\lambda dt + dW_t)\,.\end{aligned}$$

An important implication of the formula (33) for the wealth evolution is that the optimal wealth evolution is independent of the choice of bond assets, which means that it is independent of the time to maturities of the bonds in which the agents invest. A different choice of bond assets will give rise to a different volatility matrix Σ_t (recall the definition of Σ_t in (16)). We can see in the optimal wealth development (33) that the volatility matrix Σ_t no longer appears.

To visualize the optimal investment strategies, we illustrate the optimal intertemporal portfolio weights based on the estimation results of the one- and two-factor models given in Table 3. Recall that the number of the bond assets needs to equal to the number of the factors, due to the no-arbitrage argument. In Fig 5, the asset in the one-factor model is a 10-year bond and the assets in the two-factor model are chosen to be one 3-year bond and one 10 year bond. The risk aversion parameter γ goes from 4 to 100. The extreme long and short investment positions in the two-factor model can be traced back to the high correlation of the factor innovations. It leads to a high degree of dependence of Γ and therefore to a near degenerate volatility matrix Σ_t, again recall the definition (16). From the formula (32) we see that the near degenerate volatility matrix results in extreme long/short investment positions.

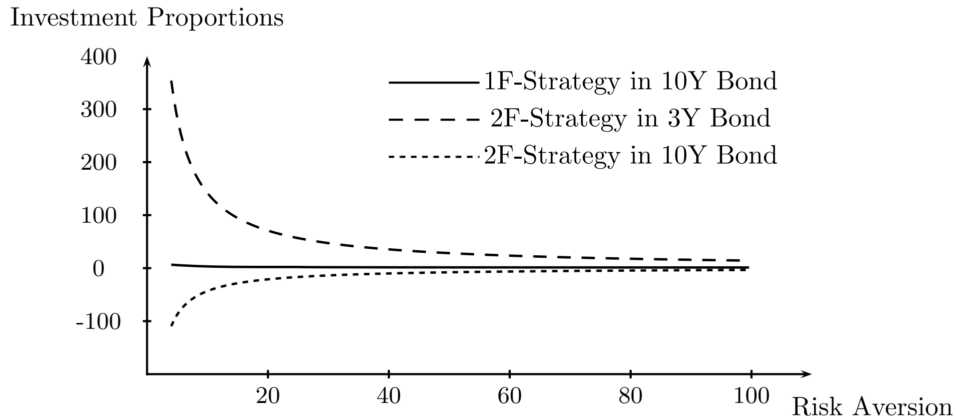


Figure 5: Optimal Investment Proportions based on the One- and Two-factor Models

4.2 Simulation Study including Measurement Errors

The analytical solution for the optimal portfolios given above is based on the exact affine term structure (2). When applying the theoretical optimal strategies to the real world we need to take account of the measurement errors that occur in the formula (28).

We develop an investment scenario and use simulation to determine the performance of the theoretical optimal strategies in the model with measurement errors. In the simulation example, we employ the two-factor model to simulate the bond price $P(t, \overline{T}_i, X_t)$ according to

$$P(t, \overline{T}_i, X_t) = e^{-A(\overline{T}-t)-\sum_{i=1,2} B_i(\overline{T}_i-t)^\top X_{it}-(\overline{T}_i-t)\epsilon_{it}}\ , \tag{34}$$

where all parameters take values from the estimation results of the two-factor model given in Table 2. The investment horizon is set to be 10 years. For the two-factor bond model, there are two bond assets in the investment set. At the initial time $t = 0$, the agents can invest in two bonds: one matures in 3 years and the other matures in 10 years. In this case we have $T_1 = 3$ and $T_2 = 10$. As time goes by, the time to maturity $T_i - t$ decreases. Once the short-term bond matures, a new 3-year bond will be introduced into the investment set immediately. So, the maturities have the time schedule show in Table 4.

	$0 \le t < 3$	$3 \le t < 6$	$6 \le t < 9$	$9 \le t \le 10$
$\overline{T}_1 =$	3	6	9	12
$\overline{T}_2 =$	10	10	10	10

Table 4: Time schedule of the bond maturities

In our simulation study we consider six different investment strategies:

- (**S1**) The first strategy is the full-information two-factor intertemporal investment strategy. It is the best theoretical investment strategy. The agents adopting this strategy possess the full information of the model of the price dynamics, which includes the number of the factors and the parameter values. The strategy is constructed by adopting the formula (32) based on the two-factor model. After elementary operations, the strategy S1 at the time t, denoted by $\alpha^*_{S1}(t)$, is given by the formula

$$\begin{aligned}\alpha^*_{S1}(t) &= \frac{1}{\gamma}\begin{pmatrix} B_1(\overline{T}_1-t) & B_1(\overline{T}_2-t) \\ B_2(\overline{T}_1-t) & B_2(\overline{T}_2-t)\end{pmatrix}^{-1}\begin{pmatrix}\Gamma_{11} & \Gamma_{21} \\ 0 & \Gamma_{22}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} \\ &\quad + \frac{1-\gamma}{\gamma}\begin{pmatrix} B_1(\overline{T}_1-t) & B_1(\overline{T}_2-t) \\ B_2(\overline{T}_1-t) & B_2(\overline{T}_2-t)\end{pmatrix}^{-1}\begin{pmatrix} B_1(10-t) \\ B_2(10-t)\end{pmatrix},\end{aligned} \tag{35}$$

  recall that $B_i(\tau) = (1-e^{-\kappa_i \tau})/\kappa_i$, and $\overline{T}_i$ for different t are given in Table 4. The agents know that the all parameter values $\mathcal{K}, \Gamma, \lambda$ and ξ_0 are given by the results of the two-factor model in Table 2.

- (**S2**) The second strategy is the full-information *mean-variance efficient* (MVE) investment strategy. Agents adopting this strategy also have the full information of the price dynamics as those adopting best theoretical investment strategy S1, but they follow the mean-variance efficient (MVE) strategy. The strategy is constructed by using the MVE portfolio, which is the first term in the formula (32) based on the two-factor model. So, this strategy can be represented by

$$\alpha^*_{S2}(t) = \frac{1}{\gamma}\begin{pmatrix} B_1(\overline{T}_1-t) & B_1(\overline{T}_2-t) \\ B_2(\overline{T}_1-t) & B_2(\overline{T}_2-t)\end{pmatrix}^{-1}\begin{pmatrix}\Gamma_{11} & \Gamma_{21} \\ 0 & \Gamma_{22}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} \tag{36}$$

where the agents also know the parameter values are given in Table 2. Recall that the strategy S2 is the best strategy when the investment environment is static. It is also in line with the conventional consideration of portfolio decisions based on the trade-off between return and risk.

- (**S3**) The third strategy is a *partial information* MVE strategy. The agents adopting this strategy have no information about the bond price dynamics. They adopt the same two-factor MVE strategy as the agents adopting S2, but they have to use the original formula given in (21), namely,
$$\alpha^*_{S3}(t) = \frac{1}{\gamma}(\Sigma_t\Sigma_t^\top)^{-1}(\mu_t - R_t\underline{\mathbf{1}}) \tag{37}$$
since they do not have information about the bond price dynamics. Their strategy to find proxies for Σ_t and μ_t is to use sample statistics of the bond daily returns. We set the learning period as one year. So, at time t the agents collect the last 250 daily bond returns for the two bonds maturing at $\overline{T}_1$ and $\overline{T}_2$ over the last year $[t, t-1]$ and subsequently calculate the sample mean and sample covariance of these daily bond returns. The sample mean minus the average riskless returns of R_t is the proxy for $(\mu_t - R_t\underline{\mathbf{1}})\Delta$ and the sample covariance matrix is the proxy for $\Sigma_t\Sigma_t^\top\Delta$.

- (**S4**) With the fourth strategy the agents keep all their wealth as money and earn the riskless instantaneous interest rate. This is just the value of the money market account and serves as a reference value.

- (**S5**) The fifth strategy is a one-factor intertemporal investment strategy. As a one-factor model investor, the agents only invest in one bond. We choose the bond maturing at the final time $T = 10$. This investment strategy is constructed by using the formula (32) based on the one-factor model, so that this strategy, denoted by $\alpha^*_{S5}(t)$ can be represented by
$$\alpha^*_{S5}(t) = \frac{1}{\gamma}\frac{\lambda_1}{B_1(10-t)\Gamma_{11}} + \frac{1-\gamma}{\gamma}\,. \tag{38}$$
We adopt the estimation results of the one-factor model given in Table 2.

- (**S6**) The last strategy is a one-factor MVE investment strategy, which can be expressed by

$$\alpha^*_{S5}(t) = \frac{1}{\gamma} \frac{\lambda_1}{B_1(10-t)\Gamma_{11}} \,. \tag{39}$$

In Table 5 we summarize the different features of the six investment strategies above.

Strategy	Type	Information	
S1	Intertemporal	The true price dynamics (the two-factor term structure model)	The best theoretical investment strategy
S2	MVE	The true price dynamics	
S3	MVE	Observed bond prices	
S4	All-money	—	
S5	Intertemporal	The one-factor model	
S6	MVE	The one-factor model	

Table 5: Six Strategies in the Simulation Study

The simulation programs are written in the programming language "GAUSS". We simulate each scenario 1,000 times for all six strategies and for two different risk aversion parameters, namely, $\gamma = 15$ and $\gamma = 30$. In order to determine the impact of the measurement errors, our simulation study includes a case with the measurement errors having $\sigma_\epsilon = 0.0006$, adopted from the estimation result of the two-factor model in Table 2, and a case without measurement errors, that is, $\sigma_\epsilon = 0$. We take the time step for 1/50, corresponding to weekly rebalancing. At the beginning of the investment period, the agents are endowed with one unit of wealth. As the criteria to evaluate performance of the strategies, we investigate both the expected final utility and the distribution of the final wealth.

Figure 6 shows the final wealth distribution for $\gamma = 15$, where the x-axis represents the final wealth and the y-axis shows the frequency. Table 6 lists the expected utility and the descriptive statistics of the final wealth. Recall that the utility is always negative for the risk aversion parameter $\gamma > 1$.

The result is quite striking because the two-factor intertemporal investment strategy S1, the best theoretical investment strategy, performs worst among all strategies, in particular, much worse than the partial information strategy S3 and the all-money-holding strategy S4. The one-factor intertemporal investment strategy S5 is the winner of the this investment competition. In the statistical summary in Table 6 we can see that the average of the final wealth by adopting best theoretical investment strategy S1 is even negative. This means, the agents are given one unit of wealth at the beginning of investment but end up with a negative outcome of wealth in average after 10 year investment following the best theoretical investment strategy!

In order to explain this outcome we trace the wealth development over the whole investment period. Figure 7 shows one typical path of the wealth development by adopting the best theoretical investment strategy S1. We can see in this figure that the trajectory of the wealth development undergoes steep falls at the time $t = 0, 3$, and 6 (year) where a new 3-year bond is introduced. At the time $t = 6$ the agents' wealth falls to a level around zero and is not able to recover subsequently before the end of the investment horizon.

In Figure 8 we plot the two-factor intertemporal investment strategy S1 through time. We can see that the time points of introduction of new bonds are also break points for the positions of the strategy S1. This suggests that the introduction of the new short-term bond causes the rapid decline in wealth.

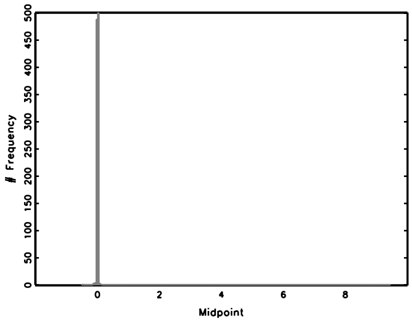
It is worth recalling here that, in the case *without* measurement errors, the wealth development under the optimal investment strategy is independent of the choice of bond securities, as illustrated in the optimal wealth dynamics (33). The impact of the introduction of the new short-term bond on the wealth development seems to be related to the existence of the measurement errors. To highlight the impact of the measurement errors we also provide one typical path of the wealth development under the theoretical investment strategies in the case without the measurement errors in Figure 9. In this figure we see indeed that the wealth development shows no breaks at the time of the introduction of the new bonds.

Figure 10 shows the final wealth distributions by adopting all six strategies for the case *without* measurement errors. From Table 6 we can see that the best theoretical intertemporal strategy S1 now indeed performs best in terms of the expected utility $\mathbf{E}[U_T]$. We also see that without measurement errors, for all six investment strategies agents are better off. The improvement of the other strategies S3 – S6 is only slight.

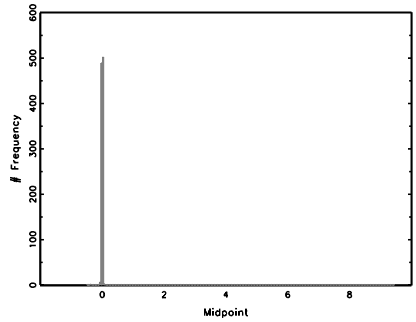
The partial information MVE strategy S3 performs on average slightly better than the "do-nothing", or the all-money-holding strategy S4. It has a higher value of expected utility and a higher average return. It should be noted that the sample mean and sample standard deviation of the bond returns cannot appropriately approximate the drift and diffusion coefficients in the model because the drift and diffusion vary with the time to maturity. Nevertheless, if one does not have other information, the strategy S3 which seeks to learn the price dynamics by observing the market prices, can still beat the all-money-holding strategy S4.

The one-factor intertemporal investment strategy S5 performs well in both cases with and without measurement errors. Its stable performance may be traced back to the fact that its holding position is not as extreme as that based on the two-factor model, as illustrated in Figure 5. On the other hand, the agents who adopt the strategy S5 do have an intertemporal model in mind. This is an informational advantage to the agents adopting MVE strategies S3 and S6.

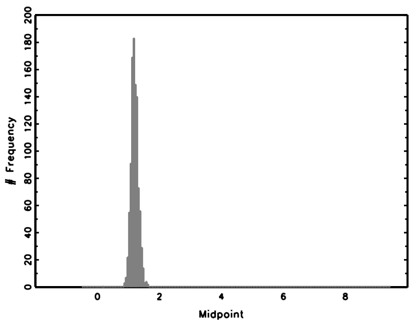
We provide also the final wealth distributions for agents with higher risk aversion $\gamma = 30$. Such agents do not take as extreme investment positions as the agents with $\gamma = 15$, so the standard variations of the final wealth distributions based on strategies S1 and S2 without measurement errors, and those based on strategies S5 and S6 with or without measurement errors, are reduced, as shown in Table 6. This more conservative attitude also makes negative final wealth a very low probability outcome by adopting strategies S1 and S2 in the presence of the measurement errors, as we can observe in the same table.



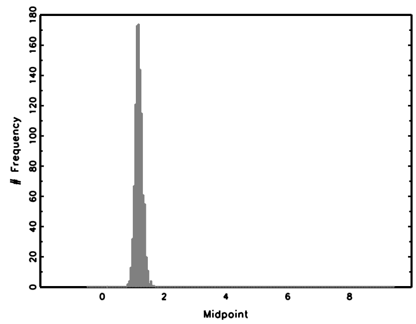
(a) S1



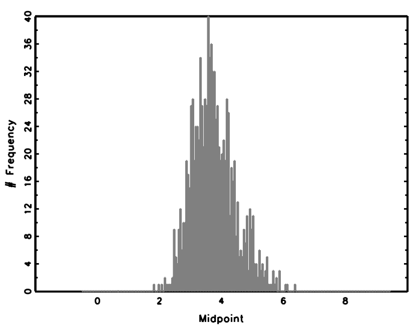
(b) S2



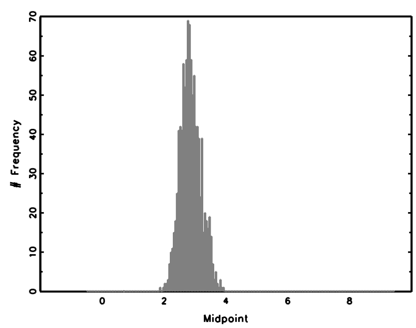
(c) S3



(d) S4



(e) S5



(f) S6

Figure 6: Final Wealth Distribution, $\gamma = 15, \sigma_\epsilon = 0.0006$

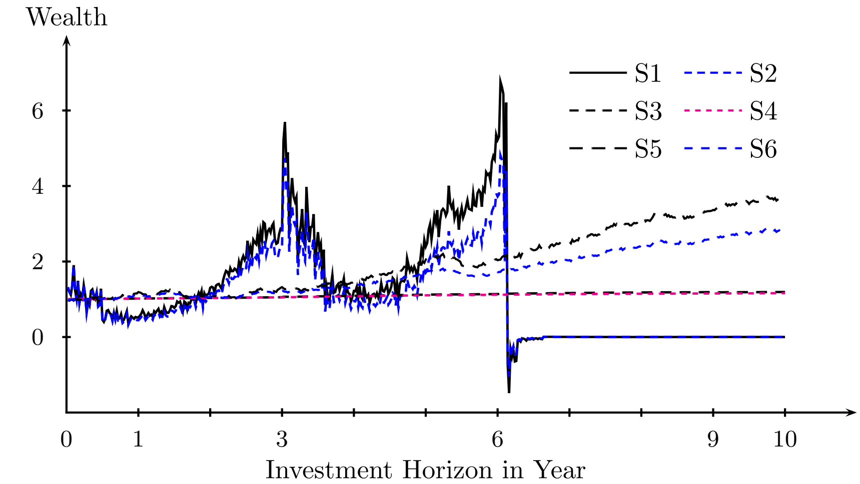


Figure 7: Wealth Development with Measurement Errors

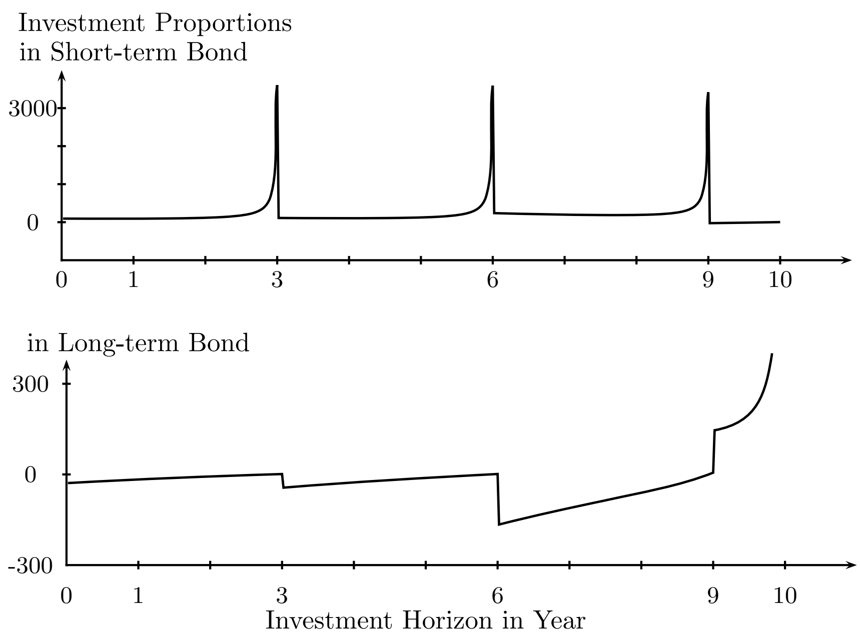


Figure 8: Theoretical Intertemporal Portfolio Proportions

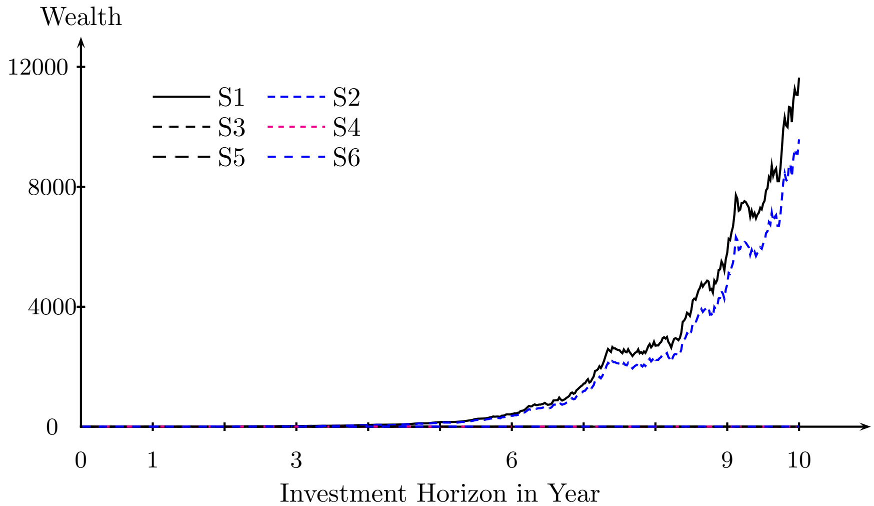
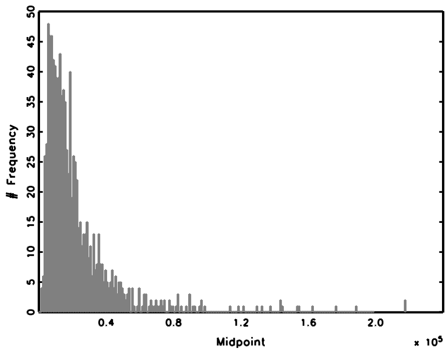
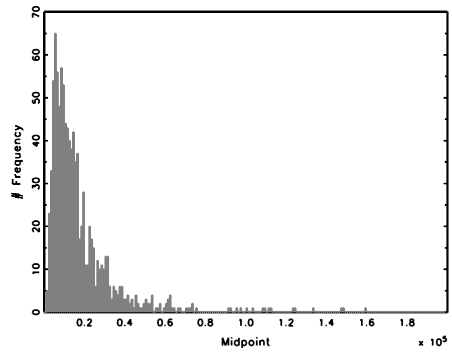


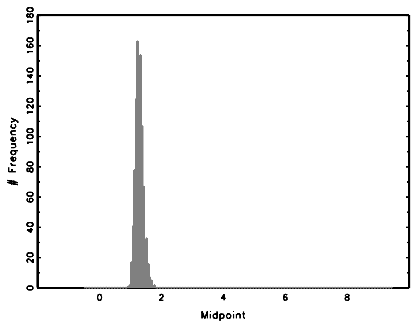
Figure 9: Wealth Development without Measurement Errors



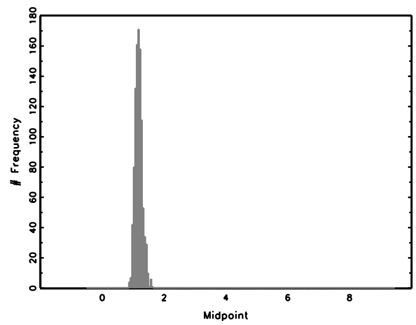
(a) S1



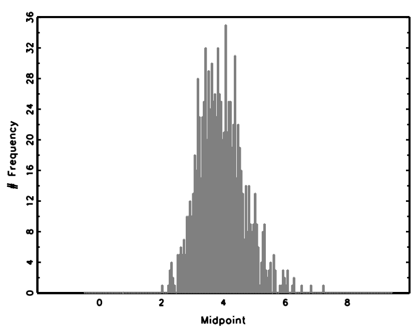
(b) S2



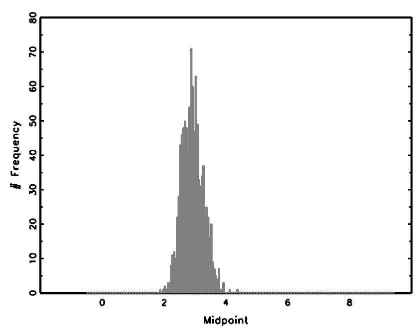
(c) S3



(d) S4

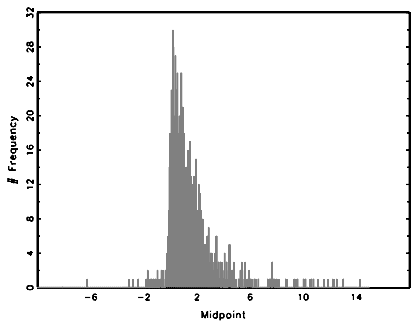


(e) S5

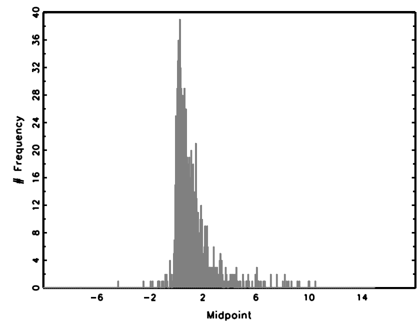


(f) S6

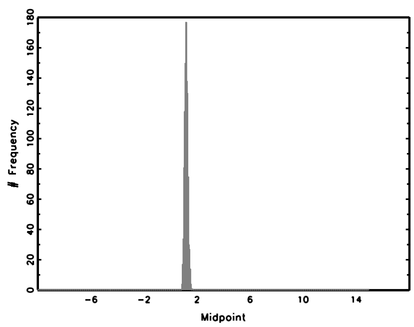
Figure 10: Final Wealth Distribution, $\gamma = 15, \sigma_\epsilon = 0$



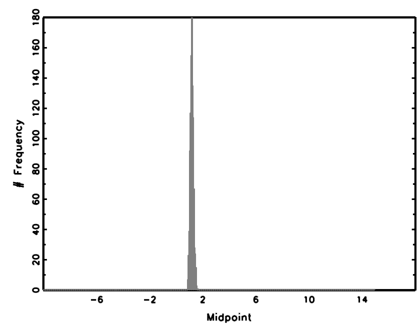
(a) S1



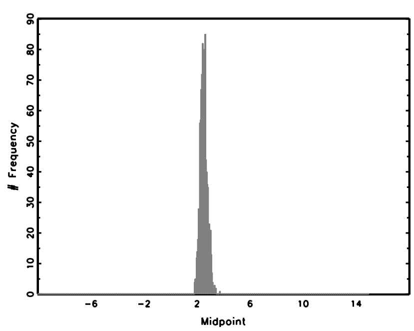
(b) S2



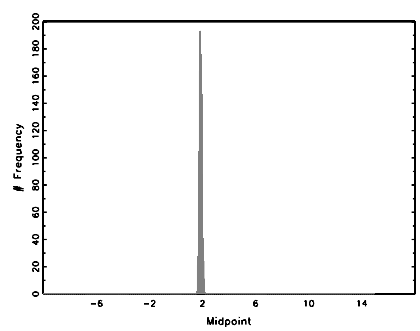
(c) S3



(d) S4

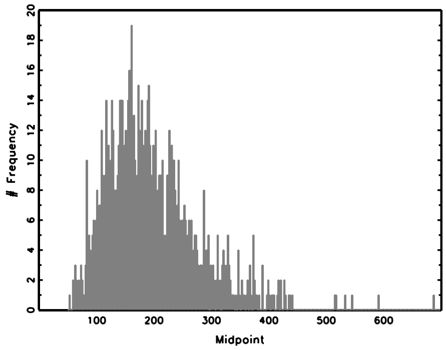


(e) S5

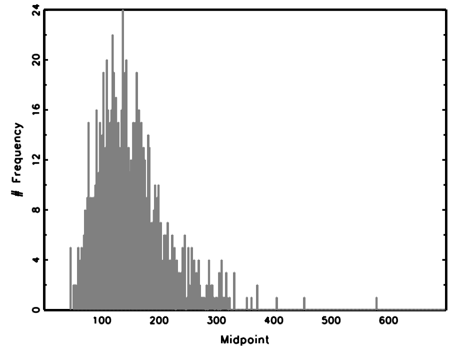


(f) S6

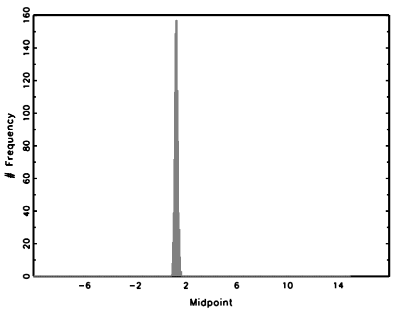
Figure 11: Final Wealth Distribution, $\gamma = 30, \sigma_\epsilon = 0.0006$



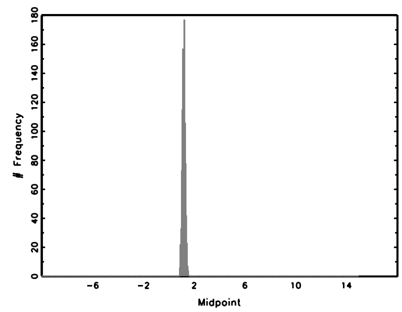
(a) S1



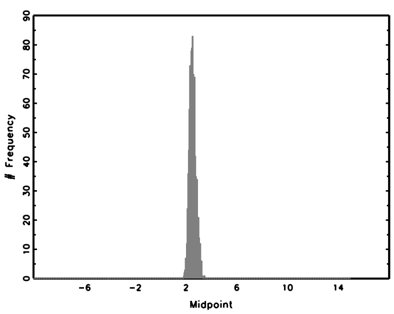
(b) S2



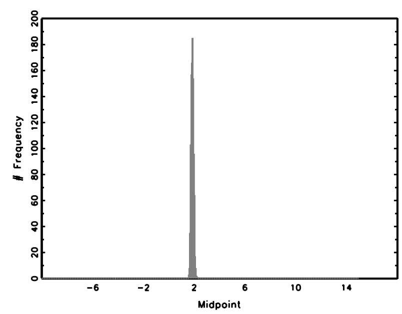
(c) S3



(d) S4



(e) S5



(f) S6

Figure 12: Final Wealth Distribution, $\gamma = 30, \sigma_\epsilon = 0$

$\gamma = 15$ $\sigma_\epsilon = 0.0006$	S1	S2	S3	S4	S5	S6
$\mathbf{E}[U(V_T)]$	$-9.33e^{86}$	$-7.35e^{86}$	-0.0153	-0.0206	$-3.85e^{-8}$	$-1.28e^{-7}$
$\mathbf{E}[V_T]$	-0.0009	-0.0007	1.2006	1.1830	3.7513	2.8585
$\sigma[V_T]$	0.0187	0.0142	0.1159	0.1189	0.7809	0.3339

$\gamma = 15$ $\sigma_\epsilon = 0.0$	S1	S2	S3	S4	S5	S6
$\mathbf{E}[U(V_T)]$	$-1.13e^{-48}$	$-5.91e^{-48}$	-0.0056	-0.0195	$-1.54e^{-8}$	$-1.07e^{-7}$
$\mathbf{E}[V_T]$	22636	17673	1.2829	1.1771	3.9325	2.9184
$\sigma[V_T]$	24141	18139	0.1278	0.1174	0.7481	0.3509

$\gamma = 30$ $\sigma_\epsilon = 0.0006$	S1	S2	S3	S4	S5	S6
$\mathbf{E}[U(V_T)]$	$-4.12e^{178}$	$-2.84e^{178}$	-0.0140	-0.0222	$-9.91e^{-12}$	$-2.85e^{-9}$
$\mathbf{E}[V_T]$	1.5921	1.1807	1.1811	1.1717	2.5047	1.8519
$\sigma[V_T]$	2.0605	1.5206	0.1180	0.1203	0.2743	0.1019

$\gamma = 30$ $\sigma_\epsilon = 0.0$	S1	S2	S3	S4	S5	S6
$\mathbf{E}[U(V_T)]$	$-3.68e^{-55}$	$-4.97e^{-52}$	-0.0057	-0.0224	$-4.24e^{-12}$	$-1.81e^{-8}$
$\mathbf{E}[V_T]$	195.12	151.74	1.2326	1.1797	2.5300	1.8595
$\sigma[V_T]$	79.73	60.46	0.1245	0.1189	0.2613	0.1030

Table 6: Portfolio Performance

5 Concluding Remarks

In this paper we have tried to develop optimal long-term bond investment strategies that can be applied in real markets. We have modelled the bond prices dynamics by employing the Gaussian sub-family of the Duffie-Kan affine model where bond yields with different time to maturities are assumed to be driven by some common factors. The factors are represented by stochastic processes and are unobservable. We use dynamic programming to set up the optimization procedures and the Feynman-Kac formula to solve the resulting HJB equation. We were able to develop the analytical solution for the intertemporal optimal strategies for the bond investments.

The model was estimated based on the data from German securities markets using the Kalman filter. Although the three-factor term structure model has the largest fitting errors and the smallest information criteria, we did not employ it because of the widely fluctuating trajectories of the filtered factors. Thus, we employed the one- and the two-factor term structure models to develop bond investment strategies.

Using the analytical solution obtained for the intertemporal optimal portfolios, we showed that the two models give very different recommendations for bond investment strategies. The best theoretical investment strategy, which is based on the estimation results of the two-factor model, tends to give a strategy with extremely large investment positions because of the high correlation of the factor innovations. With regard to this point, the results of the simulation study have revealed the fact that an investment strategy with such large positions is very vulnerable to measurement errors.

In the simulation study we simulated the bond prices based on the estimateion results of the two-factor term structure model and investigated the performance of six different investment strategies: the two-factor intertemporal strategy S1, the two-factor MVE strategy S2, the partial information MVE strategy S3, the all-money-holding strategy S4, the one-factor intertmeporal strategy S5, and the one-factor MVE strategy S6, in the scenarios with and without measurement errors. The best theoretical investment strategy S1 performs the worst among all the strategies in the presence of the measurement errors because of their large long/short positions. The partial information MVE strategy S3 performed only slightly better than the all-money-holding strategy S2 because the sample mean and variance are not a good proxy for the time-varying drift and diffusion of the bond returns.

The one-factor intertemporal strategy S5 stood out in our simulation study because of its stable and relatively good performance for both cases with and without the measurement errors. The success might be explained by two features of this strategy that come together: on the one hand, it incorporate the information of the time-changing investment environment; on the other hand, the investment positions are of a reasonable scale.

The approach of this paper can be extended in several ways in future research. For example, we could include stocks in order to study the interaction between stocks and bonds in the intertemporal asset allocation problem. Intermediate consumption has not yet been considered in this study and will be the object of future research.

6 Appendix

6.1 Proof

The characterization of the invariant transformation of the parameters has already been stated in Dai and Singleton (2000) . Here we provide a more detailed proof.

Lemma 4.1 *(Dai and Singleton (2000)) The invariant transformation of the parameters* $\big(\mathcal{K}, \theta, \Gamma, \xi_0, \xi_1\big)$ *of equations (2), (5), (9) and (10) with respect to the factor transformation*

$$X_t^{\mathcal{L}} := \mathcal{L} X_t + \overline{\Theta} \tag{40}$$

is given by

$$\big(\mathcal{L}\mathcal{K}\mathcal{L}^{-1},\ \mathcal{L}\theta + \overline{\Theta},\ \mathcal{L}\Gamma,\ \xi_0 - \xi_1^\top \mathcal{L}^{-1}\overline{\Theta},\ (\mathcal{L}^\top)^{-1}\xi_1\big)\ . \tag{41}$$

Proof
The first three invariant parameter transformation can be determined easily. We denote $\mathcal{K}^{\mathcal{L}}, \theta^{\mathcal{L}}, \Gamma^{\mathcal{L}}$ as the new parameters for the new factor dynamics

$$dX_t^{\mathcal{L}} = \mathcal{K}^{\mathcal{L}}(\theta^{\mathcal{L}} - X_t^{\mathcal{L}})dt + \Gamma^{\mathcal{L}} dW_t \quad . \tag{42}$$

Under the factor transformation (40), the new factor dynamics can be transformed into

$$\begin{aligned} dX_t^{\mathcal{L}} &= \mathcal{L} dX_t = \mathcal{L}\mathcal{K}(\theta - X_t)dt + \mathcal{L}\Gamma dW_t \\ &= \mathcal{L}\mathcal{K}\big(\theta - \mathcal{L}^{-1}(\mathcal{L}X_t + \overline{\Theta}) + \mathcal{L}^{-1}\overline{\Theta}\big)dt + \mathcal{L}\Gamma dW_t \\ &= (\mathcal{L}\mathcal{K}\mathcal{L}^{-1})\big(\mathcal{L}\theta + \overline{\Theta} - X_t^{\mathcal{L}}\big)dt + \mathcal{L}\Gamma dW_t\ . \end{aligned} \tag{43}$$

Identifying the two dynamical systems (42) and (43), we obtain

$$\begin{aligned} \mathcal{K}^{\mathcal{L}} &= \mathcal{L}\mathcal{K}\mathcal{L}^{-1}\ , \\ \theta^{\mathcal{L}} &= \mathcal{L}\theta + \overline{\Theta}\ , \\ \Gamma^{\mathcal{L}} &= \mathcal{L}\Gamma\ . \end{aligned} \tag{44}$$

Let $B^{\mathcal{L}}(\tau), A^{\mathcal{L}}(\tau)$ be the new coefficients in the yield formula (2) based on the transformed factor $X_t^{\mathcal{L}}$. The invariant transformation must satisfy two requirements. First, the bond formulas must remain invariant under the transformation, so that

$$y(t, t+\tau, X_t) = \frac{A(\tau)}{\tau} + \frac{B(\tau)^\top}{\tau} X_t \equiv \frac{A^{\mathcal{L}}(\tau)}{\tau} + \frac{B^{\mathcal{L}}(\tau)^\top}{\tau} X_t^{\mathcal{L}} \,. \tag{45}$$

Replacing the new factor $X_t^{\mathcal{L}}$ in equation (45) with its definition given in (40), we obtain for the new coefficients $B^{\mathcal{L}}(\tau)$ and $A^{\mathcal{L}}(\tau)$ the equalities

$$B^{\mathcal{L}}(\tau)^\top = B(\tau)^\top \mathcal{L}^{-1} \,, \tag{46}$$

$$A(\tau) = A^{\mathcal{L}}(\tau) + B(\tau)^\top \mathcal{L}^{-1} \overline{\Theta} \,. \tag{47}$$

The second requirement for the invariant transformation is that the new coefficient $B^{\mathcal{L}}(\tau)$ and $A^{\mathcal{L}}(\tau)$ must still satisfy the no-arbitrage equations (9) and (10) with the new parameters given in (44). That is, the coefficient $B^{\mathcal{L}}(\tau)$ must satsify

$$\frac{d}{d\tau} B^{\mathcal{L}}(\tau) = -(\mathcal{K}^{\mathcal{L}})^\top B^{\mathcal{L}}(\tau) + \xi_1^{\mathcal{L}} = -(\mathcal{L}^{-1})^\top \mathcal{K}^\top \mathcal{L}^\top B^{\mathcal{L}}(\tau) + \xi_1^{\mathcal{L}} \,.$$

Multifying $\mathcal{L}^\top$ on both sides, we obtain

$$\frac{d}{d\tau}\Big(\mathcal{L}^\top B^{\mathcal{L}}(\tau)\Big) = -\mathcal{K}^\top \mathcal{L}^\top B^{\mathcal{L}}(\tau) + \mathcal{L}^\top \xi_1^{\mathcal{L}} \,.$$

This equation can be simplified further to

$$\frac{d}{d\tau} B(\tau) = -\mathcal{K}^\top B(\tau) + \xi_1^{\mathcal{L}} \,, \tag{48}$$

due to the fact $\mathcal{L}^\top B^{\mathcal{L}}(\tau) \equiv B(\tau)$ from the equality (46).

Identifying the new differential equation (48) for $B^{\mathcal{L}}(\tau)$ with the original one (9), it turns out that the new parameter $\xi_1^{\mathcal{L}}$ must satisfy

$$\mathcal{L}^\top \xi_1^{\mathcal{L}} = \xi_1 \,. \tag{49}$$

Applying the second requirement also to the coefficient $A^{\mathcal{L}}(\tau)$, it has to satisfy the no-arbitrage condition (10) with the new parameters (44) and the new coefficient $B^{\mathcal{L}}(\tau)$, thus

$$\frac{d}{d\tau} A^{\mathcal{L}}(\tau) = \left(\mathcal{K}^{\mathcal{L}} \theta^{\mathcal{L}} - \Gamma^{\mathcal{L}} \lambda\right)^\top B^{\mathcal{L}}(\tau) - \frac{1}{2} \sum_{i,j=1}^{n} B_i^{\mathcal{L}}(\tau) B_j^{\mathcal{L}}(\tau) \Gamma_i^{\mathcal{L}} \Gamma_j^{\mathcal{L}} + \xi_0^{\mathcal{L}} \,. \tag{50}$$

We can observe that

$$\sum_{i,j=1}^{n} B^{\mathcal{L}}_i(\tau)B^{\mathcal{L}}_j(\tau)\Gamma^{\mathcal{L}}_i\Gamma^{\mathcal{L}}_j = \big(B^{\mathcal{L}}(\tau)^\top\Gamma^{\mathcal{L}}\big)\big(B^{\mathcal{L}}(\tau)^\top\Gamma^{\mathcal{L}}\big)^\top = \big(B(\tau)^\top\Gamma\big)\big(B(\tau)^\top\Gamma\big)^\top .$$

Applying this equalty and the definitions of new parameters given in (44) to the differential equation (50), then it can be rewritten further to

$$\begin{aligned} \frac{d}{d\tau}A^{\mathcal{L}}(\tau) &= (\mathcal{K}\theta + \mathcal{K}\mathcal{L}^{-1}\overline{\Theta} - \Gamma\lambda)^\top B(\tau) - \frac{1}{2}\sum_{i,j=1}^{n} B_i(\tau)B_j(\tau)\Gamma_i\Gamma_j + \xi^{\mathcal{L}}_0 \\ &= \frac{d}{d\tau}A(\tau) + (\mathcal{K}\mathcal{L}^{-1}\overline{\Theta})^\top B(\tau) + \xi^{\mathcal{L}}_0 - \xi_0 . \end{aligned} \tag{51}$$

The second equality above is obtained by using the original no-arbitrage condition (10).

Now, we differentiate both sides of (47) and then replace $\frac{d}{d\tau}B(\tau)$ by the original no-arbitrage condition (9), then we obtain

$$\begin{aligned} \frac{d}{d\tau}A(\tau) &= \frac{d}{d\tau}A^{\mathcal{L}}(\tau) + \frac{d}{d\tau}B(\tau)^\top\mathcal{L}^{-1}\overline{\Theta} \\ &= \frac{d}{d\tau}A^{\mathcal{L}}(\tau) + (-B(\tau)^\top\mathcal{K} + \xi_1^\top)\mathcal{L}^{-1}\overline{\Theta} . \end{aligned} \tag{52}$$

Identifying the two equations (51) and(52), it follows that the new parameter $\xi^{\mathcal{L}}_0$ has to satisfy

$$\xi^{\mathcal{L}}_0 = \xi_0 - \xi_1^\top\mathcal{L}\overline{\Theta} . \tag{53}$$

We note that the price of risk λ remains unchanged under the factor transformation because we keep the original factor uncertainty W_t. We recall that the price of risk is the compensation for bearing the uncertainty W_t.
□

Proof of Property 2
Because $\mathcal{K}$ is diagonal, we can solve every component of the coefficient $B(\tau)$ separately. Together with Condition (iii) in Property 1, the i-th component of $B(\tau)$ has to satisfy

$$\frac{d}{d\tau}B_i(\tau) = -\kappa_i B_i(\tau) + 1 , \tag{54}$$

from which the solution (11) follows readily by applying the integration factor $e^{\kappa_i\tau}$ and the initial condition $B_i(0) = 0$[12].

[12]Equation (54) is sloved by applying the intergrating factor $e^{\kappa_i\tau}$ and rewritting it as

$$e^{\kappa_i\tau}\big(\frac{d}{d\tau}B_i(\tau) + \kappa_i B_i(\tau)\big) = \frac{d}{d\tau}\big(e^{\kappa_i\tau}B_i(\tau)\big) = e^{\kappa_i\tau} ,$$

The solution given in (12) is simpliply obtained by subsituting the expression (11) into the terms in (10) and then integrating.
□

Proof of Property 3

The proof proceeds in two steps.

1: Change of probability measure

Let $\mathcal{P}$ denote the original, so called physical or historical, probability measure for the underlying process (5). Now, a new equivalent measure is defined by the Radon-Nikodym derivative

$$\frac{d\hat{\mathcal{P}}_s}{d\mathcal{P}_s} = \exp\Big(\frac{1-\gamma}{\gamma}\lambda^\top (W_s - W_t) - \frac{(1-\gamma)^2}{2\gamma^2}\lambda^\top \lambda (s-t)\Big) \ .$$

Using Girsanov's theorem, the shifted Brownian motion

$$d\hat{W}_t = dW_t - \frac{1-\gamma}{\gamma}\lambda dt$$

is a Brownian motion under the new measure $\hat{\mathcal{P}}$.
Inserting the shifted Brownian motion into the original underlying process (5), we obtain the stochastic differential equation

$$\begin{aligned} dX_t &= \mathcal{K}(\theta - X_t)dt + \Gamma dW_t \\ &= \Big(\mathcal{K}(\theta - X_t) + \Gamma\frac{1-\gamma}{\gamma}\lambda\Big)dt + \Gamma d\hat{W}_t \ . \end{aligned} \tag{55}$$

2: Application of the Feynman-Kac formula

By applying the Feynman-Kac formula, see the Theorem 1 in the Appendix 6.2, we obatin the result (26).
□

Proof of Property 4

Inserting the expressions for h_t into (24) and the Radon-Nikodym derivative

which is readily integrated.

in (27) into the expectation operator representation (26), we obtain

$$\Phi(t,T,x) = \mathbf{E}_{t,x}[e^{\Psi(t,T)}] \,, \tag{56}$$

where we use $\Psi(t,T)$ to denote

$$\begin{aligned}
\Psi(t,T) \;&:=\; -\frac{\delta}{\gamma}(T-t) + \frac{1-\gamma}{2\gamma^2}\lambda^\top\lambda(T-t) + \frac{1-\gamma}{\gamma}\int_t^T R_s ds \\
&\quad + \frac{1-\gamma}{\gamma}\lambda^\top(W_T - W_t) - \frac{(1-\gamma)^2}{2\gamma^2}\lambda^\top\lambda(T-t) \\
&= \; -\frac{\delta}{\gamma}(T-t) + \frac{1-\gamma}{2\gamma}\lambda^\top\lambda(T-t) + \frac{1-\gamma}{\gamma}\lambda^\top(W_T - W_t) \\
&\quad + \frac{1-\gamma}{\gamma}\xi_0(T-t) + \sum_{i=1}^n \frac{1-\gamma}{\gamma}\int_t^T X_{is} ds \,.
\end{aligned} \tag{57}$$

The second equality in equation (57) is due to the fact that

$$R_s = \xi_0 + \sum_{i=1}^n X_{is},$$

which is based on the result (3) and the identification restriction (iii) in Property 1. Note that the process X_{is} is the i-th component of the factor X_s.

Since the matrix $\mathcal{K}$ is diagonal due to the identification restriction (i) in Property 1, the underlying process (5) can be expressed componentwise as

$$dX_{is} = \kappa_i(\theta_i - X_{is})ds + \Gamma_i dW_s \,.$$

The solution of the stochastic differential equation above is given by[13]

$$X_{is} = e^{-\kappa_i(s-t)}X_{it} + \int_t^s e^{-\kappa_i(s-u)}\Gamma_i dW_u \,.$$

So, the last term of the equation (57) becomes

$$\begin{aligned}
\int_t^T X_{is} ds \;&=\; \int_t^T e^{-\kappa_i(s-t)}X_{it} ds + \int_t^T \int_t^s e^{-\kappa_i(s-u)}\Sigma_i dW_u ds \\
&= \; \frac{1}{\kappa_i}(1 - e^{-\kappa_i(T-t)})X_{it} + \int_t^T \int_u^T e^{-\kappa_i(s-u)} ds \Sigma_i dW_u \\
&= \; B_i(T-t)X_{it} + \int_t^T B_i(T-u)\Sigma_i dW_u \,.
\end{aligned}$$

[13]See Kloeden and Platen (1992) .

Using this result to rewrite equation (57), we obtain

$$\begin{aligned}\Psi(t,T) &= -\frac{\delta}{\gamma}(T-t)+\frac{1-\gamma}{2\gamma}\lambda^\top\lambda(T-t)+\frac{1-\gamma}{\gamma}\xi_0(T-t)\\ &\quad +B(T-t)X_t+\int_t^T \big(B(T-u)^\top\Sigma+\lambda^\top\big)dW_u\,.\end{aligned}$$

It is easy to see that $\Psi(t,T)$ is normally distributed with the expectation

$$\mathbf{E}\Psi(t,T) = -\frac{\delta}{\gamma}(T-t)+\frac{1-\gamma}{2\gamma}\lambda^\top\lambda(T-t)+\frac{1-\gamma}{\gamma}\xi_0(T-t)+B(T-t)X_t$$

and the variance

$$\mathbf{Var}\Psi(t,T)=\int_t^T \big(B(T-u)^\top\Sigma+\lambda^\top\big)\big(B(T-u)^\top\Sigma+\lambda^\top\big)^\top ds\,.$$

Using the well-known result concerning the expected value of the exponential of a normally distributed random variable, we obtain from (56) that

$$\Phi(t,T,x)=\mathbf{E}_{t,x}[e^{\Psi(t,T)}]=e^{\mathbf{E}\Psi(t,T)+\frac{1}{2}\mathbf{Var}\Psi(t,T)}\,,$$

which is equivalent to the expression (30) in Property 4.
□

6.2 Some Basic Results

Theorem 1 (Feynman-Kac Formula) *Let X_t be the solution of the stochastic differential equation (SDE)*

$$dX_t=\hat{F}_t dt+\hat{G}_t d\hat{W}_t \tag{58}$$

the infinitesimal generator of which is given by

$$\hat{\mathcal{D}}_t=\hat{F}_t^\top\frac{\partial}{\partial x}+\frac{1}{2}\sum_{i,j=1}^n \hat{G}_{it}\hat{G}_{jt}^\top\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\,.$$

Let h, g, and Ψ are functions with dimentionality $h:\mathbb{R}^d\times\mathbb{R}_+\to\mathbb{R}$, $g:\mathbb{R}^d\times\mathbb{R}_+\to\mathbb{R}$, and $\Psi:\mathbb{R}^d\times\mathbb{R}_+\to\mathbb{R}$. If $\Psi(x,t)$ satisfies the PDE

$$\frac{\partial}{\partial t}\Psi(x,t)+\hat{\mathcal{D}}_t\Psi(x,t)+h(x,t)\Psi(x,t)+l(x,t)=0\,, \tag{59}$$

subject to the boundary condition

$$\Psi(x,T) = \omega(x)\ , \tag{60}$$

then

$$\Psi(x,t) = \hat{\mathbf{E}}_{t,x}\Big[\omega(X_T)e^{\int_t^T h(X_s,s)ds} + \int_t^T l(X_s,s)e^{\int_t^s h(X_u,u)du}ds\Big]\ , \tag{61}$$

where $\hat{\mathbf{E}}_{t,x}$ *is the expectation operator with respect to the stochastic process* $X_s, s \geq t$ *satisfying the SDE (58) with initial position* $X_t = x$.

For the proof of the Feymann-Kac formula, see, for example, Øksendal (2003) or Korn (1997).
□

Kalman Filter

The Kalman filter is employed to estimate the model consisting of one observation equation

$$y_t = Z_t X_t + d_t + \varepsilon_t\ , \tag{62}$$

and one state equation

$$X_t = T_t X_{t-1} + c_t + R_t \eta_t\ . \tag{63}$$

The notation is adopted from Harvey(1990). For each t, the $N \times 1$-vector y_t is directly observable. On the right hand side of (62) the observations are explained by an observable component d_t and *the unobservable* state variable X_t. The state variable follows the dynamics (63). The Kalman filter can estimate the unobservable X_t based on the information/observations until t. Further details of the Kalman filter may obtained in Harvey(1990).

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